Unit-I

Probability and Random Variables

Introduction:- The primary goals of the Subject to study about the principles of random signals and to provide tools to deal with systems involving random signals.

Random Signal :-

The outcome of an experiment which cannot be predicted from its past values is known as random signal (or) non-deterministic signals.

Eg:- Examples of Random Signals

i) Now considering experiment of tossing a coin the outcome should be either Head (or) Tail.

ii) while listening to an ordinary radio resiver sum noise will occur, that noise is called "Hiss" noise.

iii) In a Television System noise shows up in the form of picture interference, that noise called snow.

Deterministic Signals :-

The outcome of an experiment to be predicted from its positive values is known as deterministic signals.

Examples of Deterministic Signals:-

i) consider an experminent to study about the "where the suns rises".

ii) The outcome of above experiment to be conducted on any day is the Same that is sun rises in east.

Probability Interduced through sets and relative frequency.

A Set is a collection of objects, the objects are also called elements of the Set. A Set is denoted by a capital letter and the elements are denoted by Lower case Letters.

Ex:- $a \in A$

* Any set is represented by two braces

$\{\}$

Countable Set :-

A Set is said to be countable if the elements can be put in one to one corrospondance with natural numbers.

ex:- A Set of natural No's

$N = \{1, 2, 3, 4\}$

A set of odd No's

uncountable set : A set is not countable then it is called uncountable set

Ex:- $C = \{0.5 \le C \le 8.5\}$

Empty set & null set : A set is said to be empty if it has no elements then set is called null set

* A is denoted by the symbol ϕ.

Ex:- $P = \{0, 0\}$ is not a null set.

Finite set :- A finite set is one that is either empty or has elements.

Ex:- $D = \{0, 0\}$ is a finite set.

$E = \{2, 4, 6, 8, 10, 12, 14\}$ is a finite set.

Infinite set :- If a set is a not a finite then is called infinite set $\{N = 1, 2, 3, 4 \ldots\}$

Countable Infinite Set :-

An infinite set having a countable elements is called countable infinite set

Ex:- The set having "N" no. of natural numbers $N = \{1, 2, 3, 4 \ldots \infty\}$

uncountable infinite set :-

An infinite set having uncountable elements is called uncountable infinite set.

Ex:- $F = \{-50 \le F \le 20.0\}$

Subset :- A set "x" is a subset of "y" element of set "y".

* It is denoted by a " $x \leq y$ ".

Equal Sets :-

A two Sets "x" and "y" are are to be equal if Both the Sets are Contained all common elements

* It is denoted by $x = y$ & $x \leq y$ and $y \leq x$

Ex :- $x = \{1, 3, 5, 7\}$, $Y = \{1, 3, 5, 7\}$

Disjoint Sets :-

If two sets "x" and "y" are Said to be disjoint Sets if they contain no common Elements.

Ex :- $x = \{1, 3, 5, 7\}$ $y = \{2, 4, 6, 8\}$

Universal Set :-

A Set which Contains all the elem of a given System is called universal set.

* It is denoted by "s".

Ex :- Rolling a die.

universal set $= \{1, 2, 3, 4, 5, 6\}$

Relative frequency :-

If we are Conducting a particular experiment more number of time by just observing the output is called Relative frequency.

* let "A" be a particular event in this Sample space "S" then the probability of event "A" is denoted by P(A).

* Suppose the Random Experiment is repeated "N" number of times if the event "A" occurs n(A) times then the probability of event "A" is denoted by relative frequency of event "A" when $n \to \infty$

$\therefore$ The probability of event "A" is Defined as

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$$

where $\frac{n(A)}{n}$ is called Relative frequency of A

n(A) means How many times a particular event occurs.

"n" means How many time the Experiment is repeated.

1. A Die is thrown 1000 times with frequencies for the outcomes 1·2·3456 is given by

outcomes	1	2	3	4	5	6
frequencies	179	150	157	149	175	190

Find the probability of outcome "1" for the experiment is repeated 1000 times.

Sol: i) P{outcome "1" for the experiment is repeated 1000 times} $= \lim_{n \to \infty} \frac{n(A)}{n}$

$= \lim_{n \to \infty} \frac{179}{1000} = \frac{179}{1000} = 0.179$

(ii) P{outcome "2" for the experiment is repeated 1000 times} $= \lim_{n \to \infty} \frac{150}{1000} = \frac{150}{1000} = 0.15$

(iii) P{outcomes "3" for the experiment is repeated is 1000 times} $= \lim_{n \to \infty} \frac{157}{1000} = \frac{157}{1000} = 0.157$

(iv) P{outcomes "4" for the experiment Repeated is 1000 times} $= \lim_{n \to \infty} \frac{149}{1000} = \frac{149}{1000} = 0.149.$

(v) P{outcomes "5" for the experiment Repeated is 1000 times} $= \lim_{n \to \infty} \frac{175}{1000} = \frac{175}{1000} = 0.175.$

(vi) P{outcome "6" for the experiment Repeated is 1000 times} $= \lim_{n \to \infty} \frac{190}{1000} = \frac{190}{1000} = 0.19$

Sum $= 0.17 + 0.15 + 0.157 + 0.149 + 0.175 + 0.19$

$= 1$

Properties of Relative frequency:-

1) $0 \leq \frac{n(A)}{n} \leq 1$, Relative frequency exist between 0 & 1.

2) If $\frac{n(A)}{n} = 0$, then event "A" never occurs.

3) If $\frac{n(A)}{n} = 1$, then event "A" occur in all "n" trails.

4) If A & B are mutually exclusive events. Any two events have no common element.

Then $\frac{n(A \cup B)}{n} = \frac{n(A)}{n} \cup \frac{n(B)}{n}$

Experiment :-

Experiment is a physical action & the Process that is observed and the result is notted..

Ex:-

1. Tossing a Coin
2. Rolling a Single die.

Sample Space(S):-

When a physical experiment is performed when the Set of all the possible outcomes of a experiment is called a Sample Space.

* It is denoted by "S"
* Sample Space is universal Set for the given Experiment.

ex:-

(i) Tossing a coil has two outcomes head(H) and Tail (T) So the Sample space S = {H, T}

(ii) Rollging a single die has 6 outcomes.

So, the Sample Space S = {1, 2, 3, 4, 5, 6}

→ Countable

Discrete Sample Space :-

A Sample space is said to be disc and infinite of the set in the sample space have infinite number of elements.

ex :- $N = \{1, 2, 3, 4 \cdots\cdots \infty\}$

→ uncountable

Continuous Sample space :-

If the sample contains an infinite number of elements with continuous values with in a given range. Then it is called continuous sample space.

ex :- $F = \{-0.5 \leq F \leq 8.5\}$ is an continuous sample space.

Events :-

An event is defined as a set of possib outcomes of an experiment

* Tossing a coin = {H, T}

mutually Exclusive events :

If any two events in an Experiment hav no common outcomes then two events are said to be mutually Exclusive events.

Ex :- 1. Draw a king card
2. Draw a Queen Card } are mutually exclusive events.

Exhaustive events :-

All possible events in a sample space are called Exhaustive events.

Ex:- Tossing a coil → Exhaustive events are two Rolling a die → Exhaustive events are Size.

Equally likely events:-

In a given experiment if one of the events is in sample it does not depend on the above another event then those events are called equally likely events.

ex:- Tossing a coil getting Heads and Tails are equally likely events.

Independent Events:-

let two events A&B in a Sample Space "S" if the probability of occurance of events "A" is not effected by the probability of occurance of event B. Then the two events are said to be two independent event.

Definition of probability:-

It is defined as Ratio of no. of favourable outcomes to the total number of Possible outcomes.

$$P = \frac{\text{No. of favourable outcomes}}{\text{Total number of possible outcomes.}}$$

Ex:- When two Dies are thrown find the Probability Sum Getting 5.

Sol:-
$$\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Total possible outcomes = 36

favourable outcomes for getting sum "5" is = 4,

Probability $p = \frac{\cancel{4}^{1}}{\cancel{36}_{9}} = \frac{1}{9}$

Axioms of probability :-

i) $P(A) \geq 0$, the probability of A is greater than or equal to zero.

The probability of event "A" is always non negative real number.

ii) $P(S) = 1$

The event "S" is an universal set and it is a sure event there the probability is always one.

$P(\emptyset) = 0$, the null set $\emptyset$ is an event and has no element it is known as impossible event and it's the probability is zero.

iii) If a Sample Space A_n has "N" element
where $n = 1, 2, 3, 4 \cdots N$

If all the events are mutually exclusive events i.e. $A_m \cap A_n = 0$ for $m \neq n$

$$P\left\{\bigcup_{n=1}^{n} A_n\right\} = \sum_{n=1}^{N} P(A_n)$$

$$P(A_1) \cup P(A_2) \cup P(A_3) \cdots \cup P(A_N) = P(A_1) + P(A_2) + P(A_3) \cdots + P(A_N)$$

The Axioms States that the probability of union of "N_1" number of mutually exclusive events is equal to the Sum of probabilites of individual events.

Properties of probability:-

1. $0 \leq P(A) \leq 1$
2. $P(A) = 1 - P(A)$
3. $P(S) = 1$
4. $P(A) < P(B)$ if $A < B$
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   If A & B are mutually exclusive Events

   Then $P(A \cap B) = 0$

   $\therefore P(A \cap B) = P(A) + P(B)$

joint probability:-

The Sample Space Consists of two events A & B which are not mutually Exclusive then the

Probability of three events occurs simultaneous is called joint probability.

For two events A & B have same com
n element for the event $A \cap B$ then few $P(A \cap B)$ is called joint probability for the two events A & B

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

if $P(A \cap B) \neq 0$ Then

$$\boxed{P(A \cup B) \leq P(A) + P(B)}$$

Conditional probability:-

~~The~~

Conditional probabilities :- P(A/B) The Conditiona Probability P(A/B) is the probability of Even 'A' occurs on the Condition that the Probability of Even 'B' is already known.

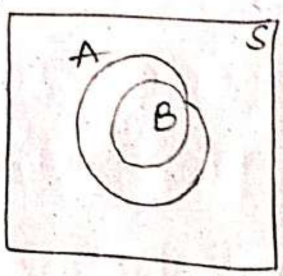
$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

Simullarly the Conditional probability P(B/A) is the probability of Even 'B' occurs on the condition the probability of Even 'A' is already known.

$$P(B/A) = \frac{P(B\cap A)}{P(A)}$$

Properties of conditional probability :

① For any two events A and B in a sample space 'S' if "B" is a subset of "A" then the probability of A/B = 1



Proof :-

Given 'B' is a Subset of 'A' as shown in figure the $P(A\cap B) = P(B)$

we know that $P(A/B) = \frac{P(A\cap B)}{P(B)}$

$$= \frac{P(B)}{P(B)} = 1$$

$\therefore P(A|B) = 1$

② If 'B' is a subset of 'A' then $P(B/A) = \frac{P(B)}{P(A)}$

Proof :-

B is a subset of 'A' as shown in the figure then $P(A \cap B) = P(B)$

we known that $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)}$

③. $P(A/B) \geq 0$ that is the conditional Probability is always a positive integer

Proof :- we know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

If $P(A \cap B)$ is greater than are equal to zero $P(A \cap B) \geq 0$ & $P(B) \geq 0$

then $P(A/B) \geq 0$

④. If two units A and B are in sample space 's' then $P(S/A) = P(S/B) = 1$ and $P(A/S) = P(A)$ and $P(B/S) = P(B)$

Proof :- Given 'A' and 'B' are in Sample space "S" then $A \leq S$ and $B \leq S$ then,

$P(A \cap S) = P(A)$; $P(B \cap S) = P(B)$

$$P(S/A) = \frac{P(A \cap S)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$P(S/B) = \frac{P(B \cap S)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$P(A/S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1} = P(A)$$

$$P(B/S) = \frac{P(B \cap S)}{P(S)} = \frac{P(B)}{1} = P(B)$$

(5) If two events A and B are mutually exclusive events then the joint events $A \cap C$ and $B \cap C$ are also mutually exclusive events then the probability of $P\left(\frac{A \cup B}{C}\right) =$

$P(A/C) + P(B/C)$

Proof :- we known that the conditional probability

$$P[(A \cup B)/C] = \frac{P[(A \cup B) \cap C]}{P(C)}$$

Now consider $P[(A \cup B) \cap C] = P(A \cap C) \cup P(B \cap C)$

By using $(x+y)c = xc + yc$

$$P((A \cup B)/C) = \frac{P(A \cap C) \cup P(B \cap C)}{P(C)}$$

$$P((A \cup B)/C) = \frac{P(A \cap C)}{P(C)} \cup \frac{P(B \cap C)}{P(C)}$$

Proof :- Given 'A' and 'B' are in sample space "S" then $A \leq S$ and $B \leq S$ then,

$P(A \cap S) = P(A)$; $P(B \cap S) = P(B)$

$$P(S/A) = \frac{P(A \cap S)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$P(S/B) = \frac{P(B \cap S)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$P(A/S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{1} = P(A)$$

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$$P((A \cup B)/C) = \frac{P(A \cap C) \cup P(B \cap C)}{P(C)}$$

$$P((A \cup B)/C) = \frac{P(A \cap C)}{P(C)} \cup \frac{P(B \cap C)}{P(C)}$$

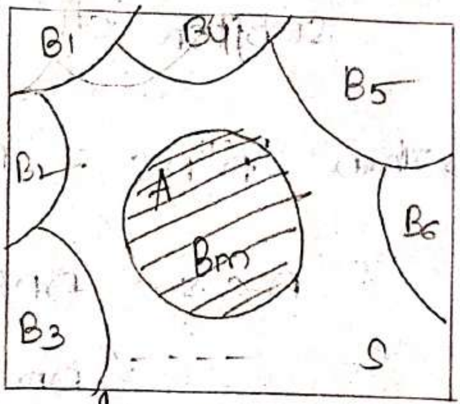
$P[(A \cap B)/C] = P(A/C) \cup P(B/C)$

$$\boxed{P[(A \cap B)/C] = P(A/C) + P(B/C)}$$

Hence proved.

Total probability Theorem:

Consider a sample space "S" has "N" number of mutually exclusive events B_n $n = 1, 2, 3 \ldots N$ such that $B_m \cap B_n = 0$ for $m \neq n$ then the probability of any event "A" defined all the sample space and can be expressed in terms of conditional probability of event B_N this probability probability of event 'A' is called total probability.



$$\therefore P(A) \sum_{n=1}^{N} P(A|B_n) P(B_n)$$

Proof :- The sample space 'S' has 'N' number of mutually exclusive B_n where $n = 1, 2, 3, \ldots N$. As shown in the figure

$\Rightarrow$ The events have the properties $B_m \cap B_n = \{\phi\}$ for $m \neq n$.

$\Rightarrow \bigcup_{n=1}^{N} B_n = S$

$B_1 \cup B_2 \cup B_3 \cup B_4 \cdots \cup B_n = S$

$\Rightarrow$ Let "A" an even "A" be defined on the Sample space "S" where the "A" is the substen of "S"

$\Rightarrow$ then "A" $\cap$ S $= A \leq S$

$$A \leq S = A \cap S = A$$

$$= A \cap \left[\bigcup_{n=1}^{N} B_n\right] = A$$

$$= A = \left[\bigcup_{n=1}^{N}\right] A \cap B_n$$

Apply probability on b.s

$$P(A) = P\left[\bigcup_{n=1}^{N} (A \cap B_n)\right]$$

$$P(A) = \bigcup_{n=1}^{N} P(A \cap B_n)$$

$$\cancel{P(A/B)} \cancel{= \frac{P(A \cap B)}{P(B)}}$$

~~From~~ the defination of join probabili the probability of $P(A \cap B_n) = P(A/B_m) P(B_n)$

$$P(A) = \bigcup_{n=1}^{N} P(A/B_m) P(B_n)$$

$$P(A) = \sum_{n=1}^{N} P(A/B_m) P(B_n)$$

$$P(A) = \bigcup_{n=1}^{N} P(A|B_n) P(B_n)$$

$$P(A) = \sum_{n=1}^{N} P(A|B_n) P(B_n)$$

Bayes Theorem :-

The Baye's theorem States that if a Sample space "S" has "N" mutually exclusive events. B_n where $n = 1, 2, 3 ---- N$ Such that $B_m \cap B_n = \{\phi\}$ for $m \neq n = 1, 2, 3 --- N$ and and event "A" is defined on the Sample Space then the conditional probability of B_n and A is given by

$$P(B_n/A) = \frac{P(A/B_n) P(B_n)}{P(A/B_1) P(B_1) + P(A/B_2) P(B_2)}$$

Proof :-

we know that the conditional probability

$$P(B_n|A) = \frac{P(A \cap B_n)}{P(A)} \text{ if } P(A) \neq 0 \quad \text{--- (1)}$$

Alternatively $P(A/B_n) = \frac{P(A \cap B_n)}{P(B_n)}$ if $P(B_n) \neq 0$ --- (2)

from eq (2)

$$P(A \cap B_n) = P(A/B_n) P(B_n) \quad \text{--- (3)}$$

one form of Bayo's theorem is obtained by Substituting eq^n (3) in eq (1)

$$P(B_n/A) = \frac{P(A/B_n)\,P(B_n)}{P(A)} \quad \text{--- (4)}$$

we know that the total probability theorem, $P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$ --- (5)

Sub eq (5) in eq (4)

$$P(B_n/A) = \frac{P(A/B_n)\,P(B_n)}{\sum_{n=1}^{N} P(A/B_n)\,P(B_n)}$$

$$P(B_n|A) = \frac{P(A/B_n)\,P(B_n)}{P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \cdots + P(A/B_N)P(B_N)}$$

Problems:-

1. In a box there are 100 resistors having resistance and tolerance values as shown in table. Let a resistor is selected from the box and assume that each resistor has the same likely hood of choosen.

Resistance	Tolerance		Total
	5%	10%	
22Ω	10	14	24
47Ω	28	16	44
100Ω	24	8	32

Event A : Draw a 47-Ω resistor

Event B : Draw a resistor with 5% tolerance

Even C : Draw a 100-Ω resistor

Find (i) Individual probability → P(A), P(B), P(C)

(ii) Joint probability → P(A∩B), P(B∩C), P(A∩C)

(iii) Conditional probability. → P(A/B) P(B/C) P(C/A)

P(B|A) P(C/B) P(A/C)

(i) Individual probabilitys

⇒ Given Event "A" is draw a 47-Ω resistor

then the probability of A $P(A) = \frac{44}{100} = 0.44$

⇒ the Event "B" draw a resistor 5%

tollerance then $P(B) = \frac{62}{100} = 0.62$

⇒ The Event "C" draw a 100-Ω resistor

the $P(C) = \frac{32}{100} = 0.32$

(ii) Joint probability

⇒ The joint Event 'AB' is draw a 47Ω resistor

with 5% Tollerance. There are 28 such

resistor.

$\therefore P(A \cap B) = \frac{28}{100} = 0.28$

⇒ The joint Event "AC" is draw a 47-Ω resistor with 100-Ω resistor.

$\therefore P(A \cap C) = 0$

Because there are no such resistor.

$P(B \cap C) = \frac{24}{100} = 0.24$.

(iii) Conditional probability :-

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.28}{0.62}$

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.28}{0.44}$

$P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{0}{0.44} = 0.$

$P(A|C) = \frac{P(A \cap C)}{P(C)} = 0$

$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{0.24}{0.32} = 0.75$

$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{0.24}{0.62} = 0.38$

Note :-

when two events A and B are independent events then $P(A \cap B) = P(A)P(B)$

$P(A \cap B \cap C) = P(A)P(B)P(C)$.

Problems:-

From a pack of 52 cards define event "A" drawing a king card event "A" and Event "B" drawing a jack & Queen cards and

Event "C" drawing a heart card then find which are independent and dependent Events.

Sol^{n}: A pack of 52 cards.

Then the probability of drawing a king card

i.e $P(A) = \frac{4}{52}$ (since there are 4 king cards).

Then the probability of drawing a jack or Queen card

i.e $P(B) = \frac{8}{52}$ (since there are 4 jack & 4 Queen cards)

Then the probability of drawing a heart card i.e. $P(C) = \frac{13}{52}$ (since there are 13 heart cards)

$P(A \cap B) = 0$

Since A & B are disjoint events.

$P(A) \cdot P(B) = \frac{4}{52} \cdot \frac{8}{52} = \frac{32}{2704} = 0.0118$

$\therefore P(A \cap B) \neq P(A)P(B)$

Then two events A & B dependent events.

$P(B \cap C) = \frac{2}{52}$ (since there are one jack and one Queen in heart card).

$P(B) \cdot P(C) = \frac{8}{52} \cdot \frac{13}{52} = \frac{104}{2704} = \frac{1}{26}$

$P: P(B \cap C) = P(B) \, P(C)$

$\therefore$ Then the two events A & B are independent events.

$$P(A \cap C) = \frac{1}{52}$$

$$P(A) \cdot P(C) = \frac{4}{52} \cdot \frac{13}{52} = \frac{52}{2704} = \frac{1}{52}$$

$\therefore P(A \cap C) = P(A) \cdot P(C)$

$\therefore$ Then the two events A & B are independent. Event.

In a factory there are four mechines, the mechines produce 10%, 20%, 30% and 40% of an items respectively. The deffective items to be produced by each machine are 5%, 4%, 3% and 2% respectively. Now an item is selected which is too be defective and what is the probabilit of defective item being from the second machine.

In a factory Given 4 machines to produce an items.

Let B_1, B_2, B_3 and B_4 are the events of to producing an items by the 4 machines

Then $P(B_1) = \frac{10}{100} = 0.1$

$P(B_3) = \frac{30}{100} = 0.3$

$P(B_4) = \frac{40}{100} = 0.4$

let "D" be the event to producing an defective item.

Given the probability of two produce items by each machine are $P(D/B_1) = \frac{5}{100} = 0.05$

$P(D/B_2) = \frac{4}{100} = 0.04$

$P(D/B_3) = \frac{3}{100} = 0.03$

$P(D/B_4) = \frac{2}{100} = 0.02$

using total probability theorem defective items to be produced by all the machines are

$$P(D) = \sum_{N=1}^{4} P(D/B_N)P(B_N)$$

$$P(D) = P(D/B_1)P(B_1) + P(D/B_2)P(B_2) + P(D/B_3)$$
$$P(B_3) + P(D/B_4)P(B_4)$$

$$P(D) = 0.05(0.1) + 0.04(0.2) + 0.03(0.3) + 0.02(0.4)$$
$$= 0.03$$

The probability of selecting defective items to be produced by second machine.

$$P(B_2/D) = \frac{P(D/B_2)P(B_2)}{P(D)} \quad [\because \text{Baye's Theorem}]$$

$= \frac{0.04(0.2)}{0.07} = 0.267$

Problems :-

① When two dice are thrown, determine the Probabilities for the following three events.

(i) $A = Sum = 7$ (ii) $B = 8 < Sum \leq 11$ (iii) $C = \{10 < Sum\}$

and also determine (iv) $P(B \cap C)$ (v) $P(B \cup C)$

Sol: When two dice are thrown

total outcomes are $6^2 = 36$

(i) For the event $A = \{sum = 7\}$, the favourable outcomes are $\{(6,1), (1,6), (5,2), (2,5), (4,3), (3,4)\}$

$\therefore P(A) = P(sum = 7) = \frac{6}{36} = \frac{1}{6} = 0.167$

(ii) For the event $B = \{8 < sum \leq 11\} = \{Sum = 9, 10 \& 11$

The favourable outcomes are $\{(3,6), (6,3), (4,5), (5,$

$(6,4), (4,6), (5,5), (5,6), (6,5)\}$

$P(B) = P\{8 < sum \leq 11\} = \frac{9}{36} = \frac{1}{4} = 0.25$

(iii) for the event $C = \{10 < Sum\} = \{Sum = 11 \& 12\}$

The favourable outcomes are $\{(5,6), (6,5), (6,6)\}$

$P(C) = \{10 < Sum\} = \frac{3}{30} = \frac{1}{12} = 0.083$

(iv) For the event $B \cap C$, the outcomes of common to both the events B and C are $\{(5,6), (6,5)\}$

$\therefore P(B \cap C) = \frac{2}{36} = \frac{1}{18} = 0.055$

(v) $P(B \cup C)$

For the event $B \cup C$, the outcomes of event are $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$$= \frac{9}{36} + \frac{3}{36} - \frac{2}{36} = \frac{10}{36} = 0.278.$$

② Two boxes are selected randomly. The first box contains 2 white balls and 3 black balls. The second box contains 3 white and 4 block balls. what is probability of drawing a white ball ?

Sol:- Assume that two boxes have equal Probability of being selected.

The probability of selecting the first box is B_1

The Probability of seleting the 2nd box is B_2

$$P(B_1) = P(B_2) = 1/2 = 0.5$$

Given the favourable event is " selecting white ball w".

The probability of selecting white ball from box B_1 is $P(w/B_1) = \frac{2}{5} = 0.4$

The probability of selecting white ball from box B_2 is $P(w/B_2) = \frac{3}{7} = 0.4285.$

using the total Probability theorem.

$$P(w) = P(w/B_1)P(B_1) + P(w/B_2)P(B_2)$$

$$= 1/2\left(\frac{2}{5} + \frac{3}{7}\right) = \frac{29}{70} = 0.4143.$$

③ A shipment of components contains of three identical boxes. One box contains 2000 components of which 25% are defective the second box has 5000 components of which 20% are defective and the third box contains 2000 components of which 600 are defective. A box is selected randomly and a component is removed at random from the box. What is the probability that this component is defective? What is the probability that it came from the second box.

Given 3 identical boxes. Let the events B_1, B_2 and B_3 are the selections of boxes. Assume that the selection is equally likely $P\{B_1\}$.

$P\{B_2\} = P\{B_3\} = 1/3 = 0.33$

Let the event "D" be "Seleting a defective component".

Given 25% of the components are defective

B_1

$$\therefore P(D|B_1) = \frac{25}{100} = 0.25$$

20% of the components are defective in B_2,

$$P(D|B_2) = \frac{20}{100} = 0.2$$

600 components are defective from 2000 in B_3

$$P(D|B_3) = \frac{600}{2000} = \frac{3}{10} = 0.3$$

using total probability theorem, the Probability of getting ~~total probability~~ a defective component is

$P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) + P(D|B_3)P(B_3)$

$= (0.25 + 0.2 + 0.3)\frac{1}{3} = \frac{0.75}{3} = 0.25$

The probability that the defective Component Comes from the Second box is

$P(B_2|D) = \frac{P(D|B_2)P(B_2)}{P(D)}$ (using Baye's theorem).

$= \frac{0.2 + 1/3}{0.25} = \frac{20}{25} = 0.267.$

④ A letter is known as to have come from either "TATANAGAR" or "CALCUTTA" on the envelope... Just two consecutive letters "TA" are visible. Find the probability that the letter has come from "CALCUTTA".

Sol: Let the event "A" be letter from TATANAGAR

let the event "B" be letter from CALCUTTA.

Assume that both events are equally likely $P(A) = P(B) = 1/2$.

Let the event "C" be the two Consecutive letters "TA".

The probability of the ~~two~~ consecutive letters "TA" being selected from TATANAGAR is

TA TA N A G AR

$$P(C/A) = \frac{2}{7}$$

The probability of the letters "TA" being selected from CALCUTTA is CALCUTTA

$$P(C/B) = 1/7$$

using Baye's theorem, the probability of the letter coming from "CALCUTTA" when the event "C" selected is

$$P(B|C) = \frac{P(C/B)P(B)}{P(C)} = \frac{P(C/B)P(B)}{P(C/A)P(A) + P(C/B)P(B)}$$

$$P(B|C) = \frac{\frac{1}{7}\times\frac{1}{2}}{\left(\frac{2}{7}\times\frac{1}{2}\right) + \left(\frac{1}{7}\times\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{7}}{\frac{2}{7}+\frac{1}{7}}$$

$$= 0.33.$$

UNIT-1

THE RANDOM VARIABLE

The Primary goals of this subject are to introduce the principles of random signals and to provide tools one can deal with systems involving such signals.

→ A random signal is a time waveform that can be characterized only in some Probabilistic manner. It can be either a desired or undesired waveform.

→ The reader has heard background hiss while listening to an ordinary broadcast radio receiver. The wave form causing the hiss. when observed on an oscilloscope, would appear as a randomly flucating voltage with time, it is undesirable.

→ Since it interferes with our ability to hear the radio program, and is called noise.

→ In a radio astronomer's receiver, noise interferes with the desired signal from outer space.

→ In a television system, noise shows up in the form of picture interference often called "snow".

→ In a sonar system, randomly generated sea sounds give rise to a noise that interferes with the desired echoes.

→ The bits in a computer bit stream appear to fluctuate randomly with time b/w the zero and one states thereby creating random signals.

→ the o/p voltage of a wind power generator would be random because wind speed fluctuates randomly.

→ The voltage from a solar detector varies randomly due to the randomness of cloud and weather conditions.

→ A set is a collection of objects. The objects are called elements of the set. A set is usually denoted by capital letter while an element is represented by a lower case letter. $a \in A$.

A set is specified by the content of two braces : { }.

→ A set is said to be countable, if the elements can be put in one-to-one correspondence with natural numbers: $A = \{1, 3, 5, 7\}$

→ If a set is not countable, it is called uncountable

$C = \{0.5 < c \leq 8.5\}$

→ A set is said to be empty, if it has no elements. The empty set is given the symbol ϕ and is often called null set.

⇒ A finite set is one that is either empty or has elements that can be counted. (or) it has a finite no. of elements.

$D = \{0.0\}$, $E = \{2, 4, 6, 8, 10, 12, 14\}$

⇒ If a set is not ~~countable, it~~ finite, it is called infinite,

An infinite set having countable elements is called countable infinite $B = \{1, 2, 3 ----\}$

An infinite set having uncountable elements is called uncountable infinite $F = \{-5.0 < f \leq 12.0\}$.

→ Experiment is any physical action or process that is observed and the result is noted Ex:- Tossing a coin, Firing a missile

⇒ A single performance of the experiment is called a trial for which there is an outcome.

→ The set of all possible outcomes in any given experiment is called the sample space. and it is given by the symbol "s".

→ An event is defined as a subset of sample space

Ex:- when die is through the sample space : $S = \{1\ 2\ 3\ 4\ 5\ 6\}$

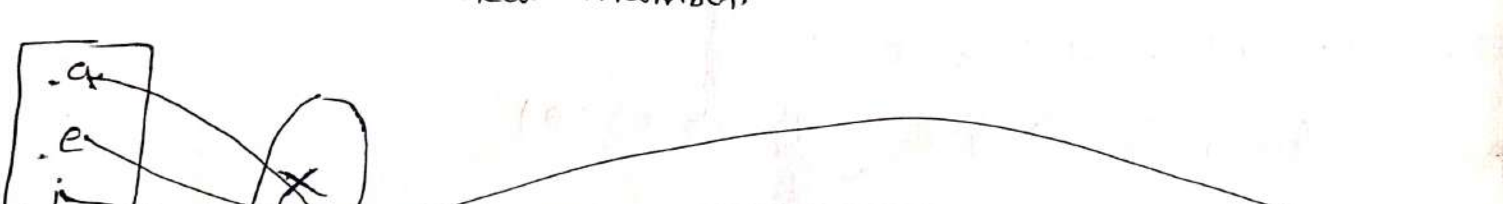
$A = \{1, 3, 5\}$ is the event of getting an odd number

$B = \{2, 4, 6\}$ " " " " even "

clearly $A \subseteq S$, $B \subseteq S$

we define a real random variable as a real function of the elements of a sample space "s". we shall represent a random variable by a capital letter (such as w, x, or y) and any particular value of the random variable by a lowercase letter (such as w, x, or y). Thus given an experiment defined by a sample space "s" with elements s. we assign to every s a real number, according to some rule and call X(s) a random variable.

Example:: consider a sample space "s" consisting of alphabets a, e, i, o, u we take a random variable 'x' in such a way that each outcome corresponds to a real number.



Here the letter "a" is the first letter in English letter and hence the real number to be assigned is one. similarly 'e' is assigned to '5', and "i" is assigned to 9 and "o" is assigned to the no of 15 and u is assigned to the number 21. Thus the random variable "x" can be consider to a function that maps all the elements of sample space onto points on the real line.

conditions for a function to be a random variable :-

The following are the necessary conditions to be satisfied by a random variable

condition I :. It is not neccessary that the sample space points map uniquely ie more than outcome may map to a simple real value

ex:. $X(d_1) = 2$ and $X(d_2) = 2$

But the random variable must not be multivalued ie every Point in the sample space must corresponds to only one real value of R.V

Condition 2: The set $\{X \le x\}$ shall be an event for any real number "x" ie The Prob of this event denoted by $P\{X \le x\}$ is equals to the sum of the probabilities of all the elementary events corresponds to $\{X \le x\}$

Eg:. $P\{X \le 4\} = P\{X=4\} + P\{X=3\} + P\{X=2\} + P\{X=1\} + P\{X=0\}$.

→. The condition we require that the probabilities of the events $\{X = \infty\}$ and $\{X = -\infty\}$ be "0"

ie $P\{X = -\infty\} = 0$ & $P\{X = \infty\} = 1$

Types of random variables:-

Basically random variables are classified into two types

1. Discreat random variable 2. continuous random variable.

and also another random variables called as mixed random "

Discrete random variable:.

In any finite intervals, if a random variable assumes only finite no. of distinct values then random variable is called discrete random variable.

Eg:. No. of telephone calls arriving at an office in a finite no. of interval of time.

Continuous random variable:-

If a random variable assigns infinite no. of uncountable values then it is called continuous random variable.

Eg:- Falling of a rain drops.

Mixed random variable:

A mixed random variable is one for which some of its values are discrete and some are continuous.

Problem: An experiment consists of a rolling a die and flipping a coin. The random variable x chosen such that

(i) a coin head (H) outcome, corresponds to positive values of x that are equal to the numbers that show up on the die.

(ii) A coin tail (T) outcome corresponds to negative values of x. That are equal in magnitude to twice the numbers that shows on the die. Map the elements of random variable "x" into points on the real line and explain.

Sol: Given that a coin head outcome corresponds to the +ve values of "x" which are equals to the number that show up on the side.

We know that the out comes of die $s = \{1, 2, 3, 4, 5, 6\}$.

The values x for head outcomes are

$(H,1) = 1$; $(H,2) = 2$; $(H,3) = 3$; $(H,4) = 4$; $(H,5) = 5$; $(H,6) = 6$;

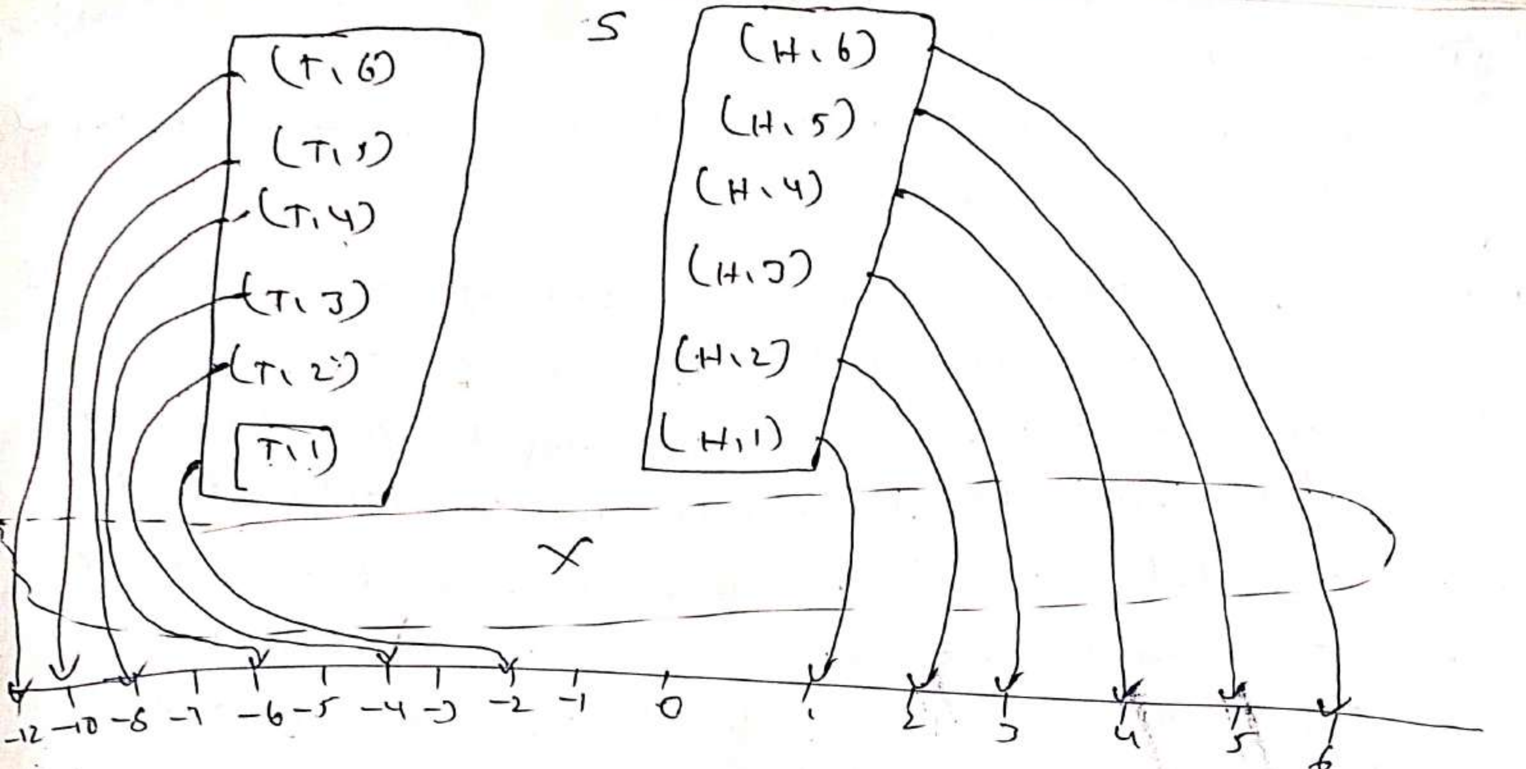
Given the coin tail (T) out come corresponds to the -ve values of x which are equal in magnitude to twice the number that shown up on a die.

The values of x for tail outcome is

$(T,1) = -2$, $(T,2) = -4$, $(T,3) = -6$, $(T,4) = -8$, $(T,5) = -10$,

$(T,6) = -12$

Explanation: Hence for a head (H) on the coin the values of random variable x lies on the +ve side of the real line. For tail on the coin the values of random variable "x" lies on the -ve side of real line.



(2)

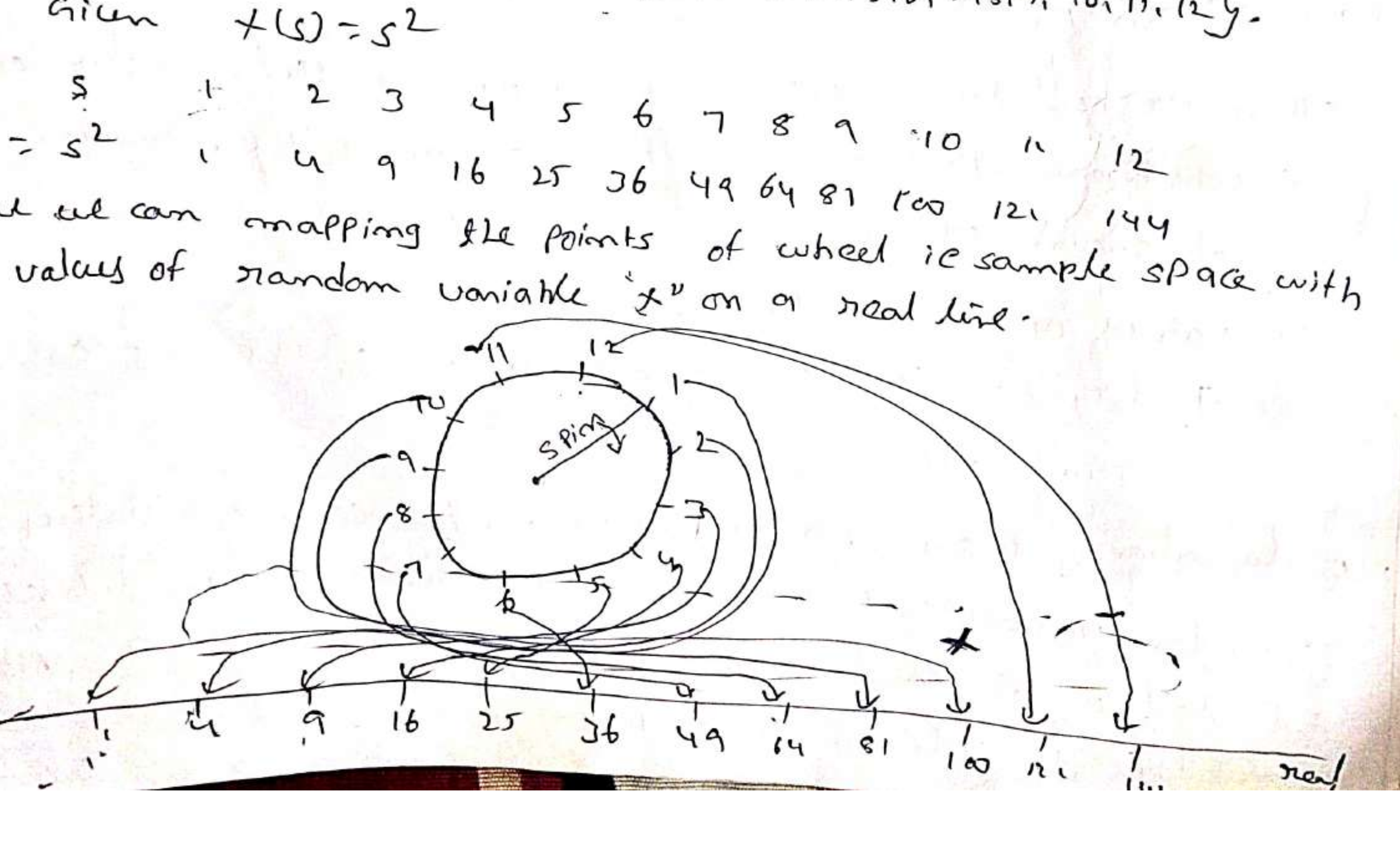
In experiment where the pointer on wheel of chance is spun. The possible outcomes are the numbers from 0 to 12 marked on the wheel. The sample space consists of the numbers on $0 \le s \le 12$. and if the random variable defined as $x = x(s) = s^2$. map the elements of random variable on real line & explain.

Given that sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Given $x(s) = s^2$

s	1	2	3	4	5	6	7	8	9	10	11	12
$x = s^2$	1	4	9	16	25	36	49	64	81	100	121	144

Here we can mapping the points of wheel ie sample space with values of random variable "x" on a real line.



Probability distribution function [cumulative distribution function] [CDF].

Probability distribution function is defined as

$$F_X(x) = P\{X \leq x\}, \quad -\infty \leq x \leq \infty$$

The Probability $P\{X \leq x\}$ is the Probability of the event $\{X \leq x\}$.
Here "X" is the function of random variable and x is the values of random variable. ie it is a function of x.

Ex:- consider a rolling die & X be the random variable which indicates the faces of die with numbers.

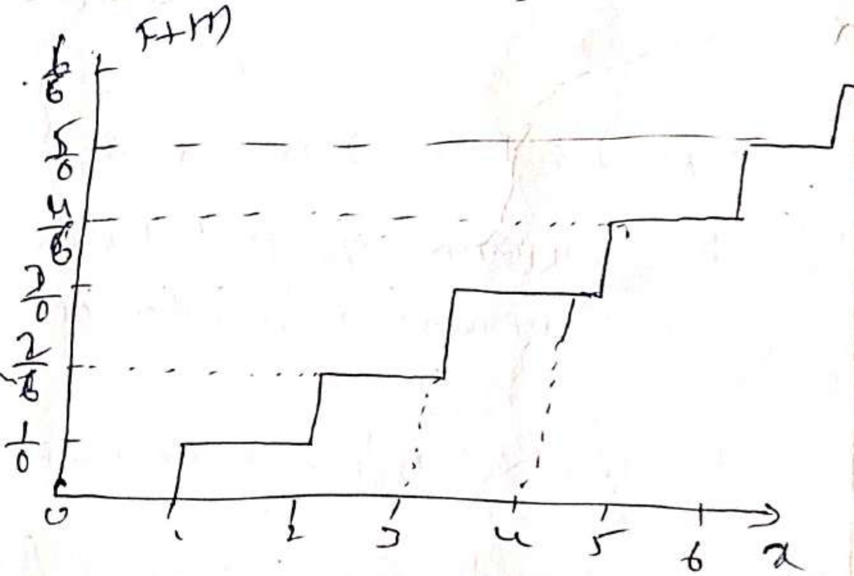
$X = x_i$	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

From the defination of distribution function $F_X(x) = P\{X \leq x\}$.

Hence $F_X(1) = P\{X \leq 1\} = P(X=1) = \frac{1}{6}$

$F_X(2) = P\{X \leq 2\} = P(X=1) + P(X=2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

The probability distribution function is in the form of stair case as shown in fig.



NOTE: The expression for distribution function for a discrete random variable X can be written as

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$$

Here $u(x)$ denotes the unit step function and is given by

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Properties of CDF:-

1. $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

Proof:- From the definition $F_X(x) = P\{X \leq x\}$

Hence $F_X(-\infty) = P\{x \leq -\infty\}$

There are no real values exists less than $-\infty$ hence $F_X(-\infty) = 0$

similarly $F_X(\infty) = P\{X \leq \infty)$

$= P\{S\} = 1$

$\therefore F_X(\infty) = 1$ & $F_X(-\infty) = 0$

2. $0 \leq F_X(x) \leq 1$

Proof:- we know that $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

hence $F_X(x)$ is lies b/w 0 and 1

$\therefore 0 \leq F_X(x) \leq 1$

3. $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

This property says that the distribution function $F_X(x)$ is non-decreasing function of "x".

4. $P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

Proof: consider $\{X \leq x_1\}$ and $\{x_1 < X \leq x_2\}$ are mutually exclusive events. $\therefore \{X \leq x_2\} = \{X \leq x_1\} + \{x_1 < X \leq x_2\}$

$P\{X \leq x_2\} = P\{X \leq x_1\} + P\{x_1 < X \leq x_2\}$

$F_X(x_2) = F_X(x_1) + P\{x_1 < X \leq x_2\}$

$\therefore P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1)$

→ $F_X(x^+) = F_X(x)$

This Property says that distribution function is continuous from right.

→ $P\{X > x\} = 1 - F_X(x)$

Proof:: consider $\{X \le x\}$ and $\{X > x\}$ are mutually exclusive events

$\therefore \{X > x\} + \{X \le x\} = S$

$P\{X > x\} + P\{X \le x\} = P(S)$

$P\{X > x\} = 1 - F_X(x).$

Problem::

A random variable "x" has the following prob function

values of x :	0	1	2	3	4	5	6
$P(x)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) the value of "k" (ii) $P(X<4)$, $P(X \ge 5)$ and $P(3 < x \le 6)$

(iii) what is the smallest values of x for which $P\{X \le x\} > \frac{1}{2}$

(iv) what is the min value of k so that prob of $P\{X \le 2\} > 0.3$

(v) Find the distribution function. $F_X(x)$.

soly::

(i) we know that the total probability is always unity

$$\sum_{i=0}^{6} P(x_i) = 1 \Rightarrow k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$k = \frac{1}{49}$$

(ii) $P\{x < 4\} = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= k + 3k + 5k + 7k = 16k = \frac{16}{49}$

$P\{x \ge 5\} = P(x=5) + P(x=6) = 11k + 13k = 24k = \frac{24}{49}$

$P\{3 < x \le 6\} = P(x=4) + P(x=5) + P(x=6) = 9k + 11k + 13k = \frac{33}{49}$

(iii) consider $P\{X \le 0\} = P(x=0) = k = \frac{1}{49}$

$P\{X \le 1\} = P(x=0) + P(x=1) = \frac{1}{49} + \frac{3}{49} = \frac{4}{49}$.

$P\{x \le 2\} = P\{x=0\} + P\{x=1\} + P\{x=2\} = \frac{9}{49} = 0.18$

$P\{x \le 3\} = P(x=0) + P(x=1) + P(x=2) + P(x=3) = \frac{16}{49} = 0.32$

$P\{x \le 4\} = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) = \frac{25}{49}$

$= 0.51 > 0.5$

Hence smallest value of x_i is 4 for $P\{X \le x\} > \frac{1}{2}$.

(iv) $P\{x \le 2\} \ge 0.3$

$P(x=0) + P(x=1) + P(x=2) > 0.3$

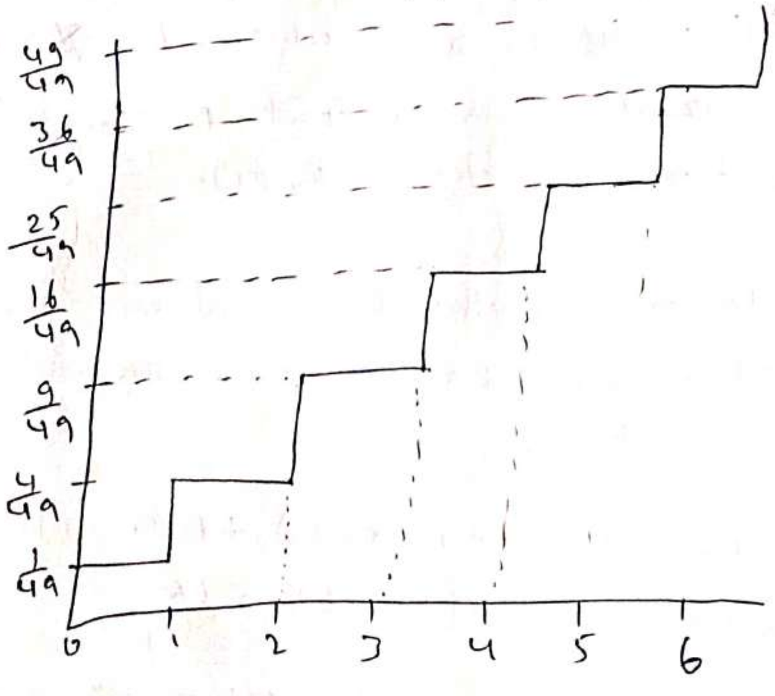
$k + 3k + 5k > 0.3$

$9k > 0.3$

$k > \frac{0.3}{9} = \frac{1}{30}$

The min value of k is $\frac{1}{30}$

x_i	:	0	1	2	3	4	5	6
$P(x_i)$	:	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$
$F_X(x)$	=	$\frac{1}{49}$	$\frac{4}{49}$	$\frac{9}{49}$	$\frac{16}{49}$	$\frac{25}{49}$	$\frac{36}{49}$	$\frac{49}{49}$



NOTE: The expression of distribution function for discrete random variable is given by $F_X(x) = \sum_{i=0}^{\infty} P(x_i)\, u(x - x_i)$.

→ The random variable "x" has the discrete variable on the set $\{-1, -0.5, 0.7, 1.5, 3\}$ the corresponding probabilities are assumed to be $\{0.1, 0.2, 0.1, 0.4, 0.2\}$. Plot its distribution function and state it is discrete or continuous.

sol:

x_i :	-1	-0.5	0.7	1.5	3
$P(x_i)$:	0.1	0.2	0.1	0.4	0.2

we know that the probability distribution function can be defined as

$$F_X(x) = P\{X \le x\}$$

$$F_X(-1) = P(X \le -1) = 0.1,$$

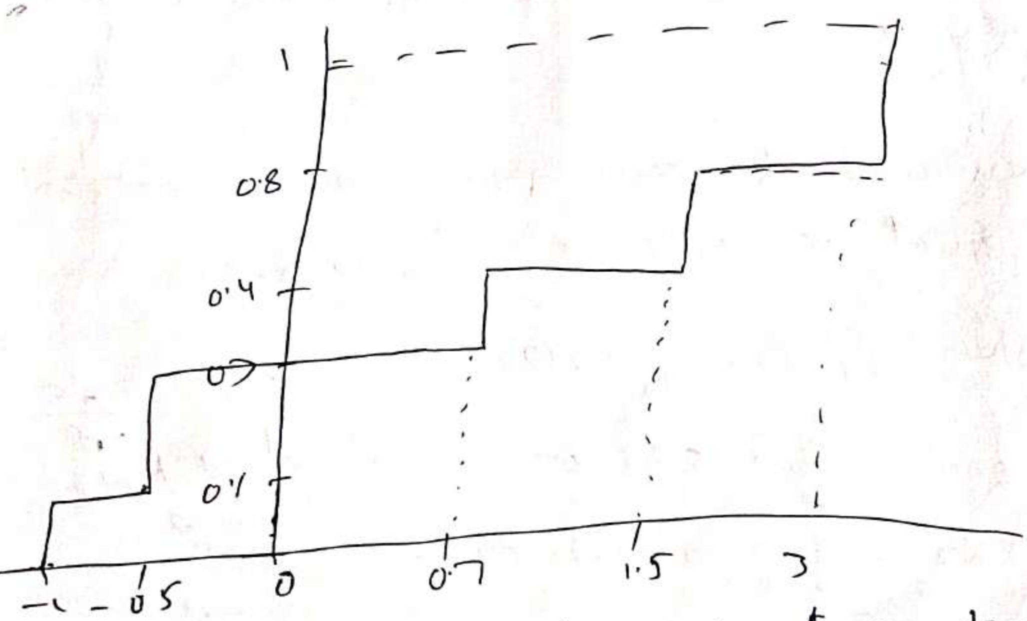
$$F_X(-0.5) = P[X \le -0.5] = 0.1 + 0.2 = 0.3$$

$$F_X(0.7) = P[X \le 0.7] = P(X=-1) + P(X=-0.5) + P(X=0.7)$$
$$= 0.1 + 0.2 + 0.1 = 0.4$$

$$F_X(1.5) = 0.4 + 0.4 = 0.8$$

$$F_X(3) = 0.8 + 0.2 = 1$$

x_i =	-1	-0.5	0.7	1.5	3
$P(x_i)$ =	0.1	0.2	0.1	0.4	0.2
$F_X(x)$ =	0.1	0.3	0.4	0.8	1



The expression for distribution function of discrete random variable

$$F_X(x) = \sum P(x_i)\, u(x - x_i)$$

$$F_X(x) = 0.1\,u(x+1) + 0.2\,u(x+0.5) + 0.1\,u(x+0.7) + 0.4\,u(x-1.5) + 0.2\,u(x-3)$$

Probability density function (P.D.F) ($f_X(x)$) :-

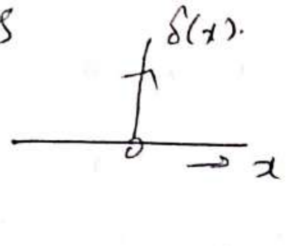
The probability density function can be defined as the derivative of distribution function $F_X(x)$. and which is denoted by $f_X(x)$ and is given by $$f_X(x) = \frac{d}{dx} F_X(x).$$

we know that distribution function is in the form of stair case form with step function, unit impulse function can be used to indicate the derivative of unit stair function ie

$$\delta(x) = \frac{d}{dx} u(x)$$

Here $\delta(x)$ is unit impulse function and $u(x)$ is unit step function, unit impulse function can be defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$



Properties of probability density function :-

Property 1 :- $f_X(x) \geq 0$ for all "x"

This property states that the probability density function $f_X(x)$ is non-negative

→ Property 2

The area under the ~~curve~~ P.D.F curve over an interval $-\infty$ to ∞ is equal to unity ie $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Proof: consider $f_X(x) = \frac{d}{dx} F_X(x)$

Take integral on both sides over an interval $-\infty$ to ∞

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x)\cdot dx = F_X(x)\Big|_{-\infty}^{\infty}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = F_X(\infty) - F_X(-\infty) = 1 - 0 = 1$$

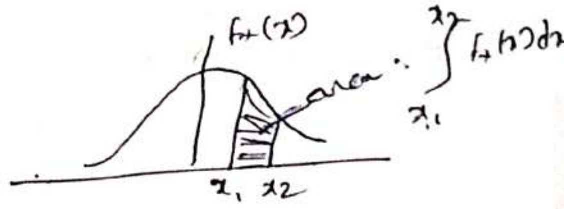
$$\therefore \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

NOTE: This property is useful for check the validity of density as well as distribution function

Property 3 :- (7)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\cdot dx = P\{X \le x\}$$

This property states that distribution function equals to the integral of density function over an interval $-\infty$ to x. (or)
The distribution function equals to the area under the P.D.F curve over an interval "$-\infty$" to 'x'.



→ (4) $$P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)\cdot dx$$

NOTE :- If "X" is a discrete random variable then

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x-x_i)$$

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x-x_i)$$

→ If "X" is a continuous random variable then

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$f_X(x) = \frac{d}{dx} F_X(x).$$

Problem :- consider an experiment of tossing four fair coins. The random variable "X" is associated with the no. of tails showing. compute and sketch the cumulative distribution function.

sol :- Give 4 fair coins are tossed then possible outcomes are $2^4 = 16$

S = { HHHH HHHT HHTH HHTT
HTHH HTHT HTTH HTTT
THHH THHT THTH THTT
TTHH TTHT TTTH TTTT }

Given X is a random variable associated with no. of tails.

$X = \{0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4\}$

$X =$	0	1	2	3	4
$P(X=x_i)$:	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F_X(x) =$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{16}{16}$

we know that $F_X(x) = P\{X \le x\}$

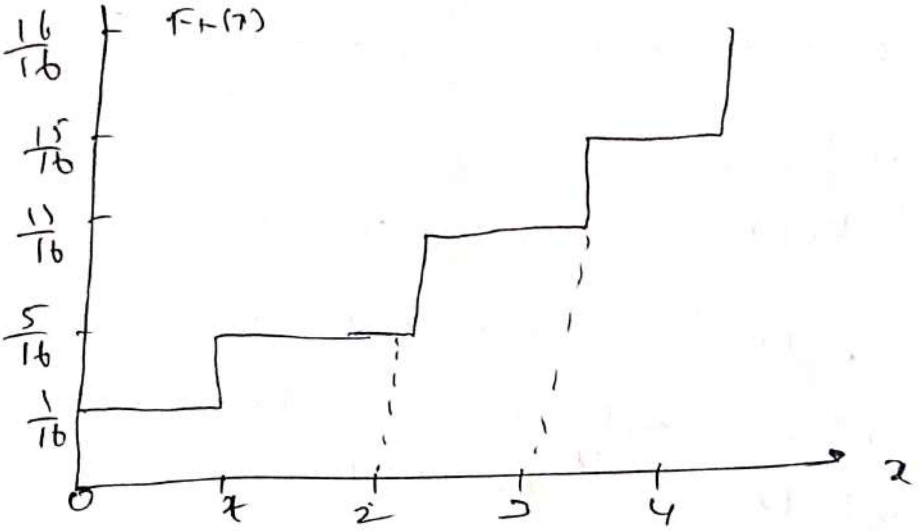
$F_X(0) = P(x \le 0) = P(x=0) = \frac{1}{16}$

$F_X(1) = P\{X \le 1\} = P(x=0) + P(x=1) = \frac{5}{16}$

$F_X(2) = P\{X \le 2\} = P\{x=0\} + P\{x=1\} + P\{x=2\} = \frac{11}{16}$

$F_X(3) = P\{X \le 3\} = P\{x=0\} + P\{x=1\} + P\{x=2\} + P\{x=3\} = \frac{15}{16}$

$F_X(4) = P\{X \le 4\} = P\{x=0\} + P\{x=1\} + P\{x=2\} + P\{x=3\} + P\{x=4\} = \frac{16}{16}$.



NOTE: From the above plot the CDF expression for discrete random variable can be written as

$$F_X(x) = \sum_{i=0}^{n} P(x_i)\, u(x - x_i)$$

$$F_X(x) = \frac{1}{16} u(x) + \frac{4}{16} u(x-1) + \frac{6}{16} u(x-2) + \frac{4}{16} u(x-3) + \frac{1}{16} u(x-4).$$

⇒ A random variable "X" has the probabilities shown in fig.

X =	-3	-2	-1	0	1	2
$P(x_i)$ =	0.2	$0.5k$	k	0.1	$0.3k$	k

(a) Find the value of k (b) Find $F_X(x)$ and $f_X(x)$ also draw the plots.

Sol:- we know that sum of all probabilities = 1

$\Sigma P(x_i) = 1$

$\Rightarrow 0.2 + 0.5k + k + 0.1 + 0.3k + k = 1$

$0.3 + 2.8k = 1$

$k = \frac{0.7}{2.8} = 0.25$

Probability function

$$F_X(x) = P\{X \le x\}.$$

$F_X(-3) = P\{X \le (-3)\} = 0.2$

$F_X(-2) = P\{X \le (-2)\} = P\{X=-3\} + P\{X=-2\} = 0.325$

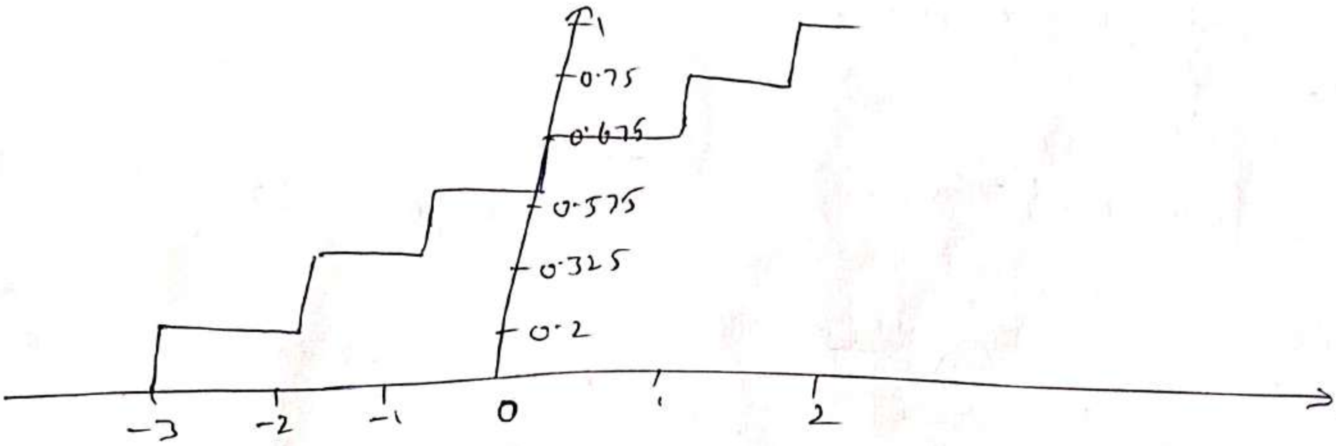
$F_X(-1) = P\{X \le -1\} = 0.2 + 0.5k + k = 0.575.$

$F_X(0) = 0.575 + 0.1 = 0.675$

$F_X(1) = 0.675 + 0.3k = 0.75$

$F_X(2) = 0.75 + 0.25 = 1$

x :	-3	-2	-1	0	1	2
$P(x_i)$:	0.2	0.125	0.25	0.1	0.075	0.25
$F_X(x)$:	0.2	0.325	0.575	0.675	0.75	1



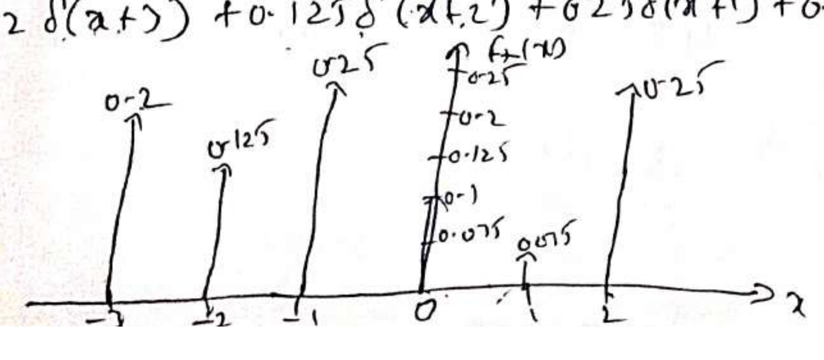
The expression of distribution function for discrete random variable X is given by $F_X(x) = \sum_{i=-3}^{2} P(x_i)\, u(x - x_i)$

$$F_X(x) = 0.2\,u(x+3) + 0.125\,u(x+2) + 0.25\,u(x+1) + 0.1\,u(x) + 0.075\,u(x-1) + 0.25\,u(x-2)$$

The expression of density function for discrete random variable is

$$f_X(x) = \sum_{i=-3}^{2} P(x_i)\,\delta(x - x_i)$$

$$f_X(x) = 0.2\,\delta(x+3) + 0.125\,\delta(x+2) + 0.25\,\delta(x+1) + 0.1\,\delta(x) + 0.075\,\delta(x-1) + 0.25\,\delta(x-2)$$



plot of $f_X(x)$

→ The CDF of a random variable Y

$F_Y(y) = \begin{cases} 1 - \exp(-0.4\sqrt{y}) & y \geq 0 \\ 0 & y < 0 \end{cases}$ calculate prob of $\{2.5 < Y \leq 6.2\}$.

Sol:. $P\{2.5 < Y \leq 6.2\} = F_Y(6.2) - F_Y(2.5)$

$F_Y(6.2) = 1 - e^{(-0.4\sqrt{y})} = 1 - e^{-0.4\sqrt{6.2}} = 0.6306$

$F_Y(2.5) = 1 - e^{(-0.4\sqrt{2.5})} = 0.46687$

$F_Y(6.2) - F_Y(2.5) = 0.1619.$

→. A random variable "x" has the distribution function $F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$. Find the prob of (a) $P\{-\infty < x \leq 6.5\}$

(b) $P\{x > 4\}$ c) $P\{6 < x \leq 9\}$.

$F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$

a) $P\{-\infty < x \leq 6.5\} = F_X(6.5) - F_X(-\infty)$

$= F_X(6.5)$

$u(6.5-n) = 1$ for $6.5 - n \geq 0$
$= 0$

$6.5 \geq n$

$n = 6.5$

$n = 6$

$F_X(6.5) = \sum_{n=1}^{12} \frac{n^2}{650} u(6.5-n)$

$= \sum_{n=1}^{6} \frac{n^2}{650} \times 1 = \frac{1}{650}\{1^2+2^2+3^2+4^2+5^2+6^2\}$

$= 0.14.$

$P\{-\infty < x \leq 6.5\} = 0.14$

b) $P\{x > 4\} = 1 - P\{x \leq 4\} = 1 - F_X(4).$

$u(4-n) = \begin{cases} 1 & \text{for } 4-n \geq 0 \\ 0 & \text{elsewhere} \end{cases}$

$F_X(4) = \sum_{n=1}^{12} \frac{n^2}{650} u(4-n).$

$n = 4$

$P\{x > 4\} = 1 - \sum_{n=1}^{4} \frac{n^2}{650}(1) = 1 - \frac{1}{650}\sum_{n=1}^{4} n^2 = 1 - 0.046$

$= 0.9538$

c) $P\{6 < x \leq 9\} = F_X(9) - F_X(6)$

$F_X(9) = \sum_{n=1}^{12} \frac{n^2}{650} u(9-n) = \sum_{n=1}^{9} \frac{n^2}{650}(1) = 0.4384$

$\therefore F_X(9) - F_X(6) = 0.2984.$

→ Determine wheather the following is valid distribution function

$$F_X(x) = \begin{cases} 1-e^{-x/2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

sol:- Given distribution function is valid if the corresponding density function is valid

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\{1-e^{-x/2}\}$$

$$= -e^{-x/2} \cdot -\frac{1}{2} = \frac{1}{2}e^{-x/2}$$

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

consider $\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} \frac{1}{2}e^{-x/2}dx$

$$= \frac{1}{2} \cdot \frac{e^{-x/2}}{-\frac{1}{2}}\Big|_0^{\infty} = -[e^{-\infty} - e^{0}] = 1$$

Hence density function is valid and also distribution function is valid.

→ Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{x}{9} + k & \text{for } 0 \leq x \leq 6 \\ 0 & \text{elsewise} \end{cases}$$

Find (i) value of k (ii) $P\{2 \leq x \leq 5\}$

sol:- Consider $\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1 = \int_0^6 \frac{x}{9} + k \, dx = 1 = \frac{x^2}{18} + kx\Big|_0^6 = 1$

$$2 + 6k = 1 \Rightarrow k = \frac{-1}{6}$$

(ii) $P\{2 \leq x \leq 5\} = \int_2^5 f_X(x)\,dx = \int_2^5 \frac{x}{9} - \frac{1}{6}\,dx = 0.666.$

→ If the prob density of random variable is given by

$$f_X(x) = \begin{cases} c\exp(-x/4) & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value "c" must have and also evaluate $F_X(0.5)$.

sol:- Given $f_X(x) = \begin{cases} c \cdot \exp(-x/4) & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1 \Rightarrow \int_0^1 c \cdot e^{-x/4}\,dx = 1 \Rightarrow \frac{c \cdot e^{-x/4}}{-1/4}\Big|_0^1 = 1$$

$$-4ce^{-x/4}\Big|_0^1 = 1 \Rightarrow -4c(e^{-1/4} - 1) = 1 \Rightarrow c = \frac{1}{4(1-e^{-0.25})} = 1.13$$

(ii) $F_X(0.5) = ?$

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$F_X(0.5) = \int_0^{0.5} 1.13 e^{-x/4} dx \Rightarrow 1.13 \frac{e^{-x/4}}{-1/4} \Big|_0^{0.5}$$

$$= -4(1.13)\left[e^{-0.5/4} - e^0\right] = \underline{0.531}.$$

→ Find a value for constant A such that $f_X(x) = 0 \quad x < -1$

is a valid prob density function. $= A(1-x^2)\cos\frac{\pi x}{2} \quad -1 \le x \le 1$

$= 0 \quad x \ge 1$

Sol:- For a valid density $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$\int_{-\infty}^{\infty} A(1-x^2)\cos\frac{\pi x}{2}\, dx = 1 \Rightarrow A\int_{-1}^{1} \cos\frac{\pi x}{2} - A\int_{-1}^{1} x^2 \cos\frac{\pi x}{2} = 1$$

$$\Rightarrow A\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_{-1}^{1} - A\left[x^2 \frac{2}{\pi}\sin\frac{\pi x}{2} - \right.$$

→ A random variable 'x' has a PDF $f_X(x) = \begin{cases} c(1-x^4) & -1 \le x \le 1 \\ 0 & \text{else} \end{cases}$

(i) find "c" (ii) find $P\{|x| < \frac{1}{2}\}$.

Sol:- we know that $\int_{-\infty}^{\infty} f_X(x)\cdot dx = 1 \Rightarrow \int_{-1}^{1} c(1-x^4)\cdot dx = 1$

$c\left[x - \frac{x^5}{5}\right]_{-1}^{1} = 1 \Rightarrow c = \frac{5}{8}$.

$f_X(x) = \frac{5}{8}(1-x^4) \quad -1 \le x \le 1$

$= 0 \quad$ else

(ii) $P\{|x| < \frac{1}{2}\} = P\{x < \frac{1}{2} \text{ or } -x < \frac{1}{2}\}$

$= P\{x < \frac{1}{2} \text{ or } x > -\frac{1}{2}\}$

$= P\{-\frac{1}{2} < x < \frac{1}{2}\}$.

$= \int_{-\frac{1}{2}}^{1/2} f_X(x)\cdot dx = \int_{-\frac{1}{2}}^{1/2} \frac{5}{8}(1-x^4)\, dx$.

$= \frac{5}{8}\left[x - \frac{x^5}{5}\right]_{-\frac{1}{2}}^{1/2} = 0.617$.

→. The P.D.F of a random variable X is given by $f_X(x) = \begin{cases} k & a \le x \le b \\ 0 & \text{else} \end{cases}$

where k is constant. (i) find value of k

(ii) Let $a=1, b=2$ calculate $P\{|x| \le c\}$ for $c = 0.5$

we know that $\int_a^b k\cdot dx = 1 \Rightarrow k\int_a^b dx = 1 \Rightarrow k = \frac{1}{b-a}$

$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}$

(ii) $a=1, b=2$

$f_X(x) = 1 \quad \text{for } 1 \le x \le 2$

$= 0 \quad$ else

$P\{|x| \le 0.5\} = P\{x \le 0.5 \text{ or } x \ge -0.5\}$

$= P\{-0.5 \le x \le 0.5\}$

$\int_{-0.5}^{0.5} 1\, dx = 0$ because $f_X(x)$ does not exists.

→ consider a prob density function $f_X(x) = a e^{-b|x|}$ where x is a random variable. whose allowable values $-\infty$ to ∞. Find (i) CDF $F_X(x)$ (ii) Relationship b/w a & b (iii) Find prob that the outcome lies b/w 1 to 2.

sol...

$$f_X(x) = a e^{-b|x|}$$

$$f_X(x) = \begin{cases} a e^{-bx} & \text{for } x \geq 0 \\ a e^{bx} & \text{for } x < 0 \end{cases}$$

(i) CDF for $x \geq 0$: $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$

$$F_X(x) = \int_{-\infty}^{0} a e^{bx} dx + \int_{0}^{x} a e^{-bx} dx$$

$$= a \cdot \frac{e^{bx}}{b}\Big|_{-\infty}^{0} + a \frac{e^{-bx}}{-b}\Big|_{0}^{x} = \frac{a}{b}[1-0) - \frac{a}{b}[e^{-bx} - 1]$$

$$= \frac{a}{b} - \frac{a}{b} e^{-bx} + \frac{a}{b} = \frac{2a}{b} - \frac{a}{b} e^{-bx} = \frac{a}{b}[2 - e^{-bx}].$$

for $x \leq 0$: $F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx = \int_{-\infty}^{x} a e^{bx} \cdot dx$

$$= \frac{a}{b} e^{bx}.$$

(ii) w.k.T the total area of density function is unity.

$$\int_{-\infty}^{0} a e^{bx} + \int_{0}^{\infty} a e^{-bx} = 1 \Rightarrow \frac{a}{b} e^{bx}\Big|_{-\infty}^{0} + \frac{a}{-b} e^{-bx}\Big|_{0}^{\infty}$$

$$= \frac{a}{b}(1) - \frac{a}{b}[0-1) = \frac{2a}{b} = 1 \Rightarrow \boxed{2a = b}$$

(iii) The prob lies b/w 1 to 2 then

$$P\{1 < x < 2\} = \int_{1}^{2} f_X(x) \cdot dx = \int_{1}^{2} a e^{-bx} dx = -\frac{a}{b}[e^{-bx}]_{1}^{2}$$

$$= \frac{a}{b}[e^{-b} - e^{-2b}].$$

→ A random variable X has the density function $f_X(x) = \begin{cases} \frac{1}{4} & -2 < x < 2 \\ 0 & \text{else} \end{cases}$

obtain (i) $P\{X < 1\}$ (ii) $P\{|x| \leq 1\}$ (iii) $P\{2X + 3 > 5\}$ $P\{2x > 2\} = P\{x > 1\}$

sol (i) $P\{x < 1\} = \int_{-2}^{1} f_X(x)\, dx = \frac{3}{4}$.

(ii) $P\{|x| \leq 1\} = P\{-1 < x < 1\} = \int_{-1}^{1} \frac{1}{4} dx = \frac{1}{2}$.

(iii) $P\{x > 1\} = \int \frac{1}{4} dx = \frac{1}{4}$

→ A continuous random variable "x" has a PDF for $f_X(x) = 3x^2$ $0 \le x \le 1$ (TT)
Find i) a & b such that (i) $P\{x \le a\} = P\{x > a\}$ (ii) $P\{x > b\} = 0.05$

sol... $P\{x \le a\} = P\{x > a\}$

$$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx \Rightarrow x^3 \Big|_0^a = x^3 \Big|_a^1 \quad \cancel{} \tfrac{1}{2}$$

$$a^3 = \frac{1}{2} \Rightarrow a = \sqrt[3]{\frac{1}{2}}.$$

$$P\{x > b\} = \int_b^1 3x^2 = x^3 \Big|_b^1 = 0.5 \Rightarrow 1 - b^3 = 0.05$$

$$b^3 = 1 - 0.05 = 1 - \frac{1}{20} = \frac{19}{20}$$

$$b = \left(\frac{19}{20}\right)^{1/3}$$

→ A random variable x has a density function $f_X(x) = k \cdot \frac{1}{1+x^2}$ determine the value of k. and also obtain distribution function.

sol W.K.T

$$\int_{-\infty}^{\infty} k \cdot \frac{1}{1+x^2} = 1$$

$$k \cdot \tan^{-1} x \Big|_{-\infty}^{\infty} = 1 \Rightarrow k\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

$$k = \frac{1}{\pi}$$

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\cdot dx = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi} \tan^{-1} \Big|_{-\infty}^{x} = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right].$$

→ If the random variable x that can assume any value b/w $x=2$ & $x=5$ has a density function $f_X(x) = k(1+x)$ find $P\{x < 4\}$

sol

$$f_X(x) = \begin{cases} k(1+x) & \text{for } 2 < x < 5 \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\cdot dx = 1 \Rightarrow k\int_2^5 (1+x)\,dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1$$

$$k\left[1 + \frac{25}{2}\right] = k = \frac{2}{27}$$

(i) $$P\{x < 4\} = \int_2^4 f_X(x)\cdot dx$$

$$= \int_2^4 \frac{2}{27}(1+x)\,dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4$$

$$= \frac{16}{27}.$$

Examples of Density and distribution functions :-

The general " " " " are

→ Gaussian density function (or) normal density function
→ uniform " "
→ Exponential " "
→ Rayleigh " "
→ Binomial " "
→ Poissons " "

NOTE :- In the above first four are continuous random variable and last two are discrete random variable.

Gaussian density function :-

A random variable X is called gaussian then its gaussian prob density function of a random variable is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \text{ for all } 'x' \quad -(1).$$

Here $\sigma_x > 0$ and $-\infty < a_x < \infty$ are real constants.

σ_x = standard deviation

a_x = Mean of the random variable

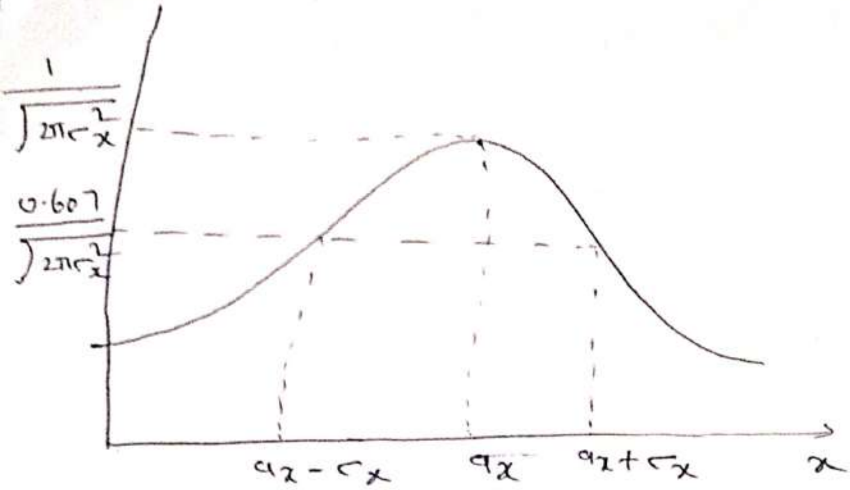
σ_x^2 = Varience of the " "

The gaussian probability distribution function is $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

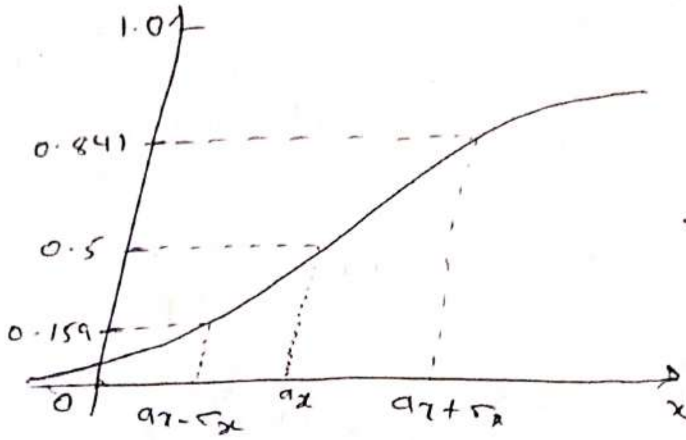
$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx \quad -(2).$$

This integral has no known closed form solution and must be evaluated by using numerical or approximation method.

we could develop a set of tables of $F_X(x)$ for various x with a_x and σ_x as parameters. This approach has limited value because there is an infinite no. of possible combinations of a_x and σ_x. which requires an infinite no. of tables.



(a) Gaussian density function



Fig(b) Gaussian distribution function.

A random variable which satisfies the gaussian ~~random variable~~ density function is called Gaussian random variable.

From fig(a)

The Gaussian density function is in Bell shape and is symmetrical about the mean. The maximum value of $f_x(x)$ is $\frac{1}{\sqrt{2\pi\sigma_x^2}}$ occurs at $x = a_x$.

The function decreases to 0.607 times of its maximum value at $x = a_x - \sigma_x$ and $x = a_x + \sigma_x$.

→ From fig(b), The distribution function $F_x(x)$ is 0.5 at $x = a_x$, 0.15 at $x = a_x - \sigma_x$ and 0.841 at $x = a_x + \sigma_x$.

Normalised Gaussian distribution Function

If the Gaussian distribution function has $a_x = 0$ and $\sigma_x = 1$ then it is called normalized gaussian distribution function. It is denoted by $F(x)$ and is given by

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} \quad \text{for } x \geq 0$$

For a negative value of x we use the $F(-x) = 1 - F(x)$.

To get the generalised distribution function $F_x(x)$ in terms of normalized distribution function $F(x)$ is consider $F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$

let $\frac{x-a_x}{\sigma_x} = u \Rightarrow \frac{1}{\sigma_x} \cdot dx = du$

Then $F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-a_x}{\sigma_x}} e^{-u^2/2} du = F(u) = F\left(\frac{x-a_x}{\sigma_x}\right)$

$$\boxed{F_X(x) = F\left(\frac{x-a_x}{\sigma_x}\right)}$$

NOTE:: The function $F(x)$ can be evaluated by using Q-function approximation. Then $F(x) = 1 - Q(x)$.

Q-function is defined as $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-x^2/2} dx$

The Q-function can be approximated as

$$Q(x) \simeq \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad x \geq 0$$

where a and b are constants

This approximation has been found to give minimum absolute relative error for any $x \geq 0$ when $a = 0.339$ and $b = 5.510$

$\therefore$ $$Q(x) \simeq \left[\frac{1}{0.661x + 0.339\sqrt{x^2+5.51}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}} \quad \text{for } x \geq 0.$$

This approximation is said to equal the true value of $Q(x)$ within a minimum absolute error of 0.27%. for any $x \geq 0$.

Applications::

The gaussian density is the most important of all densities and it enters into nearly all areas of science and engineering.

→ Especially in electronics and communication system the distribution of noise signal either internally generated or externally generated is exactly match with Gaussian prob.

Hence it is possible to elliminate the noise completely by knowing its behaviour using gaussian density function.

Problems:-

Find the prob of the event $\{x \le 5.5\}$ for a gaussian random variable having $a_x = 3$ and $\sigma_x = 2$.

sol:: Given that Gaussian random variable and $a_x = 3$, $\sigma_x = 2$

$$P\{x \le 5.5\} = F_x(5.5) = F\left[\frac{x - a_x}{\sigma_x}\right] = F\left[\frac{5.5-3}{2}\right] = 0.8944$$

→ Assume that the height of the clouds above the ground at some location is a gaussian random variable x with $a_x = 1830$ and $\sigma_x = 460$ m. We find the prob that clouds will be higher than 2750 m

sol: Given that Gaussian random variable with $a_x = 1830$ m and $\sigma_x = 460$ m. The prob of clouds will be higher than 2750 m is given by

$$P\{x > 2750\} = 1 - P\{x \le 2750\} = 1 - F_x(2750) = 1 - F\left[\frac{2750 - 1830}{460}\right]$$

$$= 1 - F(2)$$

$$= 1 - 0.9773 = 0.0227.$$

The normalised distribution function can also approximated by using Q-function

$$Q(x) = \left[\frac{1}{0.661x + 0.339\sqrt{x^2 + 5.51}}\right] \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

W.K.T $F(x) = 1 - Q(x)$

$$F(2) = 1 - Q(2)$$

$$= 1 - 0.02281 = 0.9772$$

$$P\{x \ge 2750\} = 1 - F(2) = 1 - 0.9772 = \cancel{0.02773} \; 0.0228$$

→ For a gaussian random variable with $a_x = 0$ and $\sigma_x = 1$ what is the prob of $P\{x > 2\}$.

sol:: Given g.r.v with $a_x = 0$, & $\sigma_x = 1$

$$P\{x > 2\} = 1 - P\{x \le 2\} = 1 - F_x(2) = 1 - F\left(\frac{2-0}{1}\right) = 1 - F(2)$$

$$= 0.0227.$$

Uniform density function :-

The uniform probability density function is defined as

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{else} \end{cases}$$ where a & b are real constants and $-\infty < a < \infty$ & $b > a$.

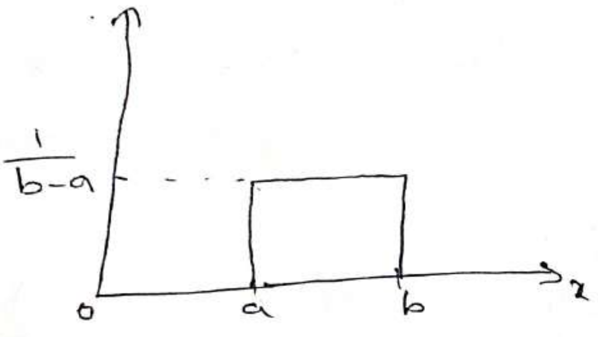
The uniform distribution function is $F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx$

$$= \int_{a}^{x} \frac{1}{b-a} dx = \frac{1}{b-a}(x-a)$$

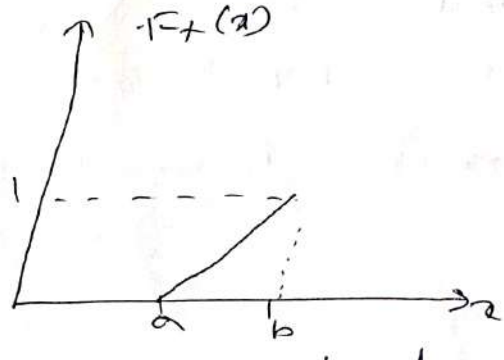
$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } b \le x \end{cases}$$

$x > a$

$x \ge b$



density function



uniform distribution function.

→ A particularly important application is in the quantization of signal samples prior to encoding in digital commun systems. Quantization amounts to "rounding off" the actual sample to the nearest of a large no of discrete "quantum levels". The errors introduced in the round-off process are uniformly distributed.

Exponential density function :-

The " prob " " is defined as

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

where a, b are real constants and $-\infty < a < \infty$ and $b > 0$

The distribution function is $F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx$

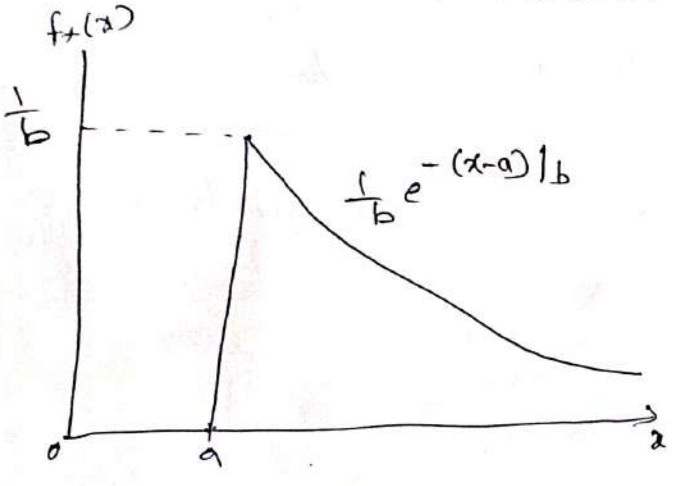
$$= \int_{a}^{x} \frac{1}{b} e^{-(x-a)/b} \, dx$$

$$= \frac{1}{b} \left[\frac{e^{-(x-a)}}{-\frac{1}{b}} \right]_a^x$$

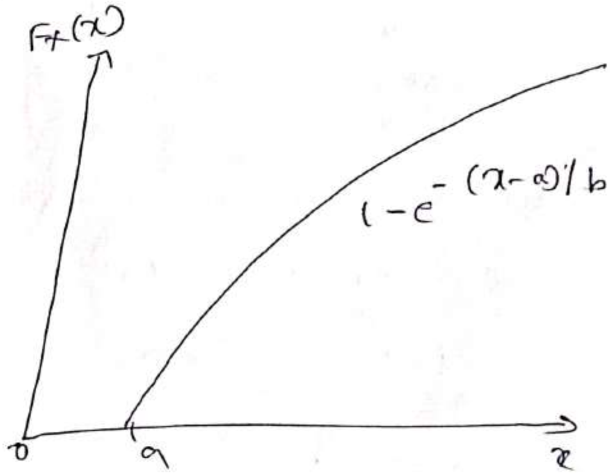
$$= - e^{-(x-a)/b} + 1$$

$$\therefore \; F_X(x) = \begin{cases} 1 - e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

The exponential density is useful in describing raindrop sizes when a large no. of rainstorm measurements are made.



Exponential density function



distribution function

Rayleigh probability density function:-

The " " " " is defined as

$$f_X(x) = \begin{cases} \frac{2}{b}(x-a) e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

Here a & b are real constants and $-\infty < a < \infty$ and $b > 0$

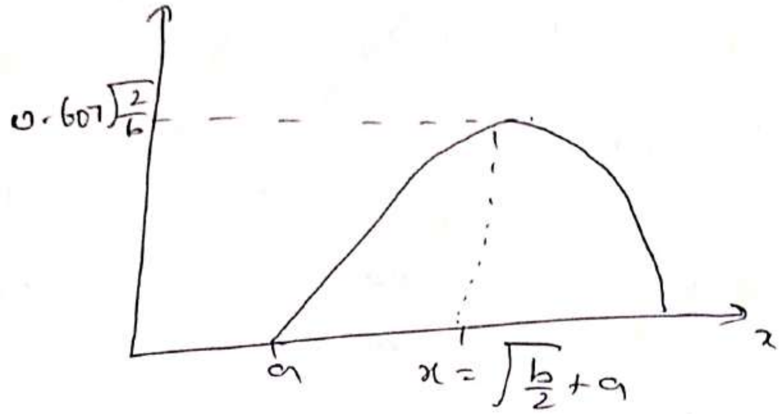
The distribution function $F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx$

$$= \int_{a}^{x} \frac{2}{b} (x-a) e^{-(x-a)^2/b} \, dx$$

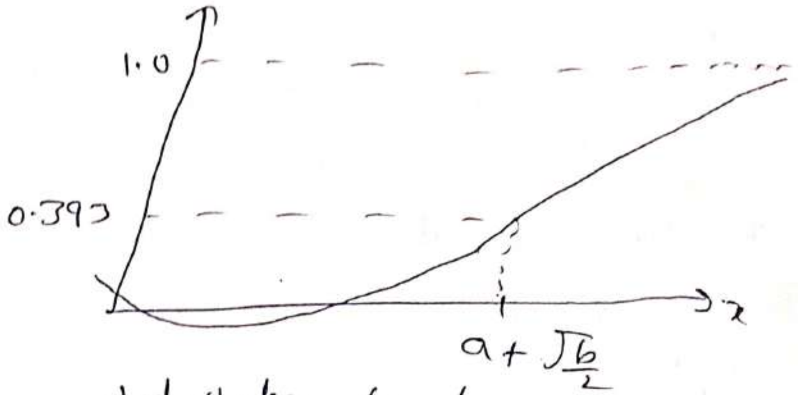
let $\left(\frac{x-a}{b}\right)^2 = u \Rightarrow dx \cdot \frac{2(x-a)}{b} = du$

$$= \int_0^u e^{-u} du = \frac{e^{-u}}{-1}\Big|_0^u = e^{-u}+1 = 1-e^{u} = 1-e^{-(x-a)^2/b}$$

$$F_X(x) = \begin{cases} 1-e^{-(x-a)^2/b} & x>a \\ 0 & x<a \end{cases}$$



Rayleigh density function



distribution function.

→ Derive the maximum value of a Rayleigh density function.

sol:- we know that the Rayleigh density function

$$f_X(x) = \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}} \quad \text{for } x \geq a$$
$$= 0 \quad x < a$$

The maximum value occurs at $\frac{d}{dx} f_X(x) = 0$

$$e^{-\frac{(x-a)^2}{b}} + (x-a)\left[e^{-\frac{(x-a)^2}{b}} \frac{d}{dx}\left(\frac{x-a}{b}\right)^2\right] = 0$$

$$(x-a)\, \cancel{e^{-\frac{(x-a)^2}{b}}} \cdot \frac{2(x-a)}{b} = \cancel{e^{-\frac{(x-a)^2}{b}}}$$

$$2(x-a)^2 = b$$
$$(x-a)^2 = \frac{b}{2}$$
$$x = a+\sqrt{\frac{b}{2}}$$

The maximum value of Rayleigh density function occurs at

$$x = a+\sqrt{\frac{b}{2}}.$$

The maximum value of $f_X(x)$ at $x = a + \sqrt{\frac{b}{2}}$ is

$$= \frac{2}{b}\left(a+\sqrt{\tfrac{b}{2}}-a\right)\left(e^{-\frac{(a+\sqrt{\frac{b}{2}}-a)^2}{b}}\right)$$

$$= \frac{2}{b}\sqrt{\frac{b}{2}}\, e^{-1/2} = \sqrt{\frac{2}{b}}\, e^{-1/2} = 0.607\sqrt{\frac{2}{b}}.$$

Derive the maximum value of Gaussian density function:

sol: w.k.T $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$ for all x

$$\frac{d}{dx} f_X(x) = 0 \Rightarrow \frac{1}{\sqrt{2\pi\sigma_x^2}} \frac{d}{dx} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} = 0$$

$$e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}\left[-\frac{1}{2\sigma_x^2}\frac{d}{dx}(x-a_x)^2\right] = 0$$

$$e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} = 0 \qquad 2(x-a_x) = 0$$

$$\boxed{x = a_x}$$

The max value is $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} = \frac{1}{\sqrt{2\pi}\cdot\sqrt{\sigma_x^2}}$

→ The Rayleigh density describes the envelope of one type of noise when passed through a bandpass filter. It also is important in analysis of errors in various measurement systems.

Binomial density function:- Let us consider an experiment having only two possible outcomes. Let the experiment be a tossing a fair coin. ~~The prob~~ Let the falling of head corresponds to success and falling of tail corresponds to failure. Let the probability of success be "p" and the prob of failure be "q". [$q = 1-p$]. such that $p+q=1$. The experiment is repeated "N" trails then the Binomial prob density ~~function~~ density function of a discrete R.V. "x" is defined as

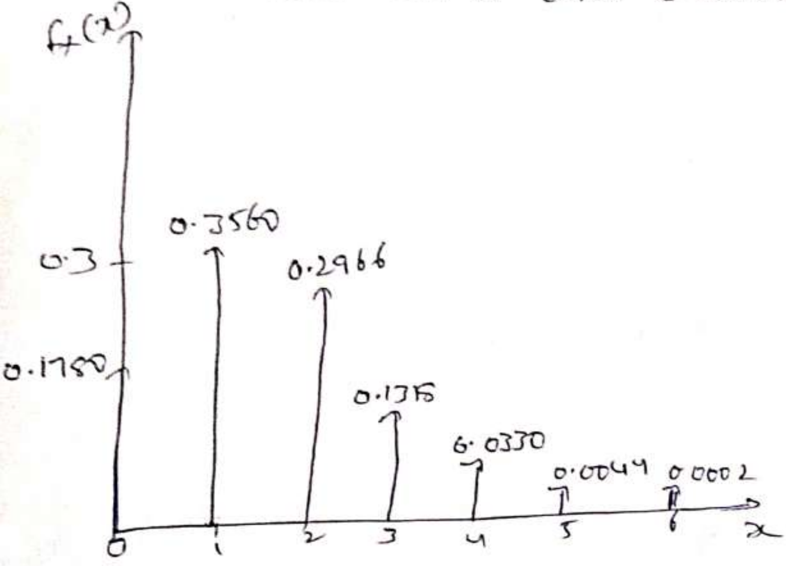
$$f_X(x) = \sum_{K=0}^{N} ({}^{N}C_K)\, p^K (1-p)^{N-K}\, \delta(x-K) \quad -(1)$$

and the binomial distribution is

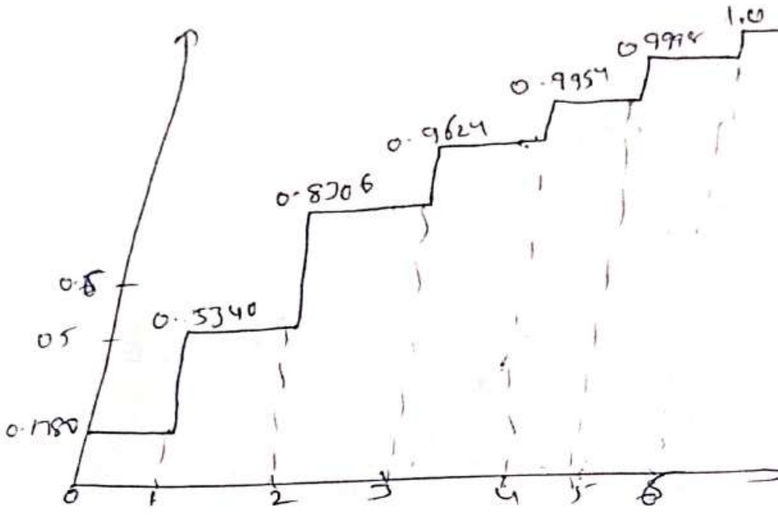
$$F_X(x) = \sum_{k=0}^{N} \underbrace{(N_{C_K})\, p^K (1-p)^{N-K}}_{\text{This is obtained by Bernoulli's trails}} u(x-k) \quad -②$$

where N_{CK} is equal to $\frac{N!}{k!(N-k)!}$

The binomial density and distribution functions for $N=6$ & $P=0.25$.



(a) Density function



(b) distribution function.

Applications :-

The Binomial density function can be used in the following applications.

1. Bernoullies Trail experiment
2. It applies to many games of chances.
3. Detection problems in sonar and radar
4. Many experiments having only two possible outcomes on a given trail.

Poisson's probability density function :-

The " " " " for a discrete random variable X is given by $f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^K}{k!} \delta(x-k) \quad -①$

The Poisson's prob distribution is

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k) \quad -②$$

The Poisson distribution is approximated function for Binomial distribution for $n \to \infty$ and $P \to 0$, Here the constant $b = NP$.

Applications:

→ It is mostly applied to counting type problems. it discribes
→ The no. of Telephone calls made during a period of time.
→ The " " defective elements in a given samples
→ The no. of electrons emitted from a cathode in a given time interval.

NOTE: If the time interval is "T" sec and the average rate of counting the items is λ then in the distribution b is given by

$$\boxed{b = \lambda T}$$

Problem:

Assume Automobile arrives at a Gasoline station are Poisson and occurs at an average rate of 50 per hour. The station has only one gasoline pump. If the all the cars are assume to require 1 minute to obtain fuel. what is the Prob that a waiting line will occurs at the pump?

Sol: Given automobile arrives at a gasoline station are Poisson
Average rate $\lambda = 50$ per hour, Time duration $T = 1$ minute

$$b = \lambda T = \frac{50}{60} \times 1 = \frac{5}{6}$$

A waiting line occurs if two or more cars arrive in one minute duration

The Prob. of waiting line equals to $P\{K \geq 2\}$.

$$= 1 - P\{K \leq 1\} = 1 - F_X(1)$$

w.k.T $\quad F_X(x) = e^{-b} \sum_{K=0}^{\infty} \frac{b^K}{K!} u(x-K)$

$$= e^{-\frac{5}{6}} \sum_{K=0}^{\infty} \frac{\left(\frac{5}{6}\right)^K}{K!} u(1-K)$$

$$= e^{-\frac{5}{6}} \sum_{K=0}^{1} \frac{\left(\frac{5}{6}\right)^K}{K!} u(1-K)$$

$$= e^{-\frac{5}{6}} \cdot \frac{\left(\frac{5}{6}\right)^0}{0!} + e^{-\frac{5}{6}} \frac{\left(\frac{5}{6}\right)^1}{1!} = e^{-\frac{5}{6}} + e^{-\frac{5}{6}} \cdot \frac{5}{6} = 0.7967.$$

$$1 - F(x) = 0.2032.$$

Conditional distribution & density function :-

Let A & B are events the conditional probability of event A given B is $P(A/B) = \frac{P(A \cap B)}{P(B)}$

Conditional distribution function :-

Let A is an event $\{X \leq x\}$, for a random variable "x" then the conditional distribution function of X, when event B is known is denoted by $F_X(x/B)$ & is defined as

$$F_X(x/B) = P\{X \leq x/B\} = \frac{P\{X \leq x \cap B\}}{P(B)} \quad -①.$$

Here probability of $P\{X \leq x \cap B\}$ indicates the probability of joint event $\{X \leq x \cap B\}$

NOTE : For discrete random variable

$$F_X(x/B) = \sum_{i=1}^{N} P(x_i/B)\, u(x - x_i) \quad -②.$$

Properties :- The properties of conditional distribution function $F_X(x/B)$ are given by

→ $F(-\infty/B) = 0$ → $F_X(\infty/B) = 1$ → $0 \leq F_X(x/B) \leq 1$

→ $F_X(x_1/B) \leq F_X(x_2/B)$ if $x_1 < x_2$

→ $P\{x_1 < X < x_2/B\} = F_X(x_2/B) - F_X(x_1/B)$

→ $F_X(x^+/B) = F_X(x/B)$.

NOTE :- we observe that except conditional event B all properties are similar to the probability distribution function.

Conditional density function :-

conditional density function of a random variable is defined as derivative of C.D.F mathematically it can be represented as

$$f_X(x/B) = \frac{d}{dx} F_X(x/B) \quad -①.$$

For discrete random variable X

$$f_X(x|B) = \sum_{i=1}^{N} P(x_i|B)\,\delta(x-x_i) \quad -(2)$$

Properties:: conditional density function $f_X(x|B)$ has the following properties

1. $f_X(x|B) \geq 0$ (non -ve quantity)
2. $\int_{-\infty}^{\infty} f_X(x|B)\,dx = 1$
3. $F_X(x|B) = \int_{-\infty}^{x} f_X(x|B)\,dx$
4. $P\{x_1 < x < x_2 / B\} = \int_{x_1}^{x_2} f_X(x|B)\,dx$.

Problem:: In an experiment there are 2 boxes each box contains balls as shown on table. The event is to select a box randomly and a ball from the selected box. The probability of selecting the first box is 0.3 then find out (i) The conditional prob distribution & density functions (ii) The prob distribution & density functions (iii) plot the graphs for the above

Ball colour (X)	Boxes 1 (B_1)	2 (B_2)	Total
Red (x_1)	10	50	60
Blue (x_2)	20	40	60
white (x_3)	80	120	200
	110	210	

Sol:: Let box 1 is B_1, Box 2 is B_2

Given $P(B_1) = 0.3$

we know that $P(B_1) + P(B_2) = 1$

$P(B_2) = 0.7$

probability of selecting a red ball from B_1 balls $P(x_1|B_1) = \frac{10}{110} = \frac{1}{11}$

" " " " Blue " " " " $P(x_2|B_1) = \frac{20}{110} = \frac{2}{11}$

" " " " white " " " " $P(x_3|B_1) = \frac{8}{11}$

" " " " red " " B_2 balls $P(x_1|B_2) = \frac{50}{210} = \frac{5}{21}$

" " " " blue " " " " $P(x_2|B_2) = \frac{40}{210} = \frac{4}{21}$

" " " " white " " " $P(x_3|B_2) = \frac{12}{21}$

Now the conditional distribution defined as

$$F_X(x|B_1) = \sum_{i=1}^{N} P(x_i|B_1)\,u(x-x_i) = \sum_{i=1}^{3} P(x_i|B_1)\,u(x-x_i).$$

$$= P(x_1|B_1)u(x-1) + P(x_2|B_1)u(x-2) + P(x_3|B_1)u(x-3)$$

$$= \frac{1}{11}u(x-1) + \frac{2}{11}u(x-2) + \frac{8}{11}u(x-3).$$

$$F_X(x|B_2) = \sum_{i=1}^{N} P(x_i|B_2)\,u(x-x_i) = \sum_{i=1}^{3} P(x_i|B_2)\,u(x-x_i).$$

$$= P(x_1|B_2)u(x-1) + P(x_2|B_2)u(x-2) + P(x_3|B_2)u(x-3)$$

$$= \frac{5}{21}u(x-1) + \frac{4}{21}u(x-2) + \frac{12}{21}u(x-3).$$

$$f_X(x|B_1) = \sum_{i=1}^{3} P(x_i|B_1)\,\delta(x-x_i)$$

$$= P(x_1|B_1)\delta(x-1) + P(x_2|B_1)\delta(x-2) + P(x_3|B_1)\delta(x-3).$$

$$= \frac{1}{11}\delta(x-1) + \frac{2}{11}\delta(x-2) + \frac{8}{11}\delta(x-1).$$

$$f_X(x_i|B_2) = P(x_1|B_2)\delta(x-1) + P(x_2|B_2)\delta(x-2) + P(x_3|B_2)\delta(x-3)$$

$$= \frac{5}{21}\delta(x-1) + \frac{4}{21}\delta(x-2) + \frac{12}{21}\delta(x-3).$$

Probability distribution function is

$$F_X(x) = \sum_{i=1}^{M} P(x_i)\,u(x-x_i)$$

$$= P(x_1)u(x-1) + P(x_2)u(x-2) + P(x_3)u(x-3)$$

$$P(x_1) = P(x_1|B_1)P(B_1) + P(x_1|B_2)P(B_2) = \frac{1}{11}(0.3) + \frac{5}{21}(0.7) = 0.193$$

$$P(x_2) = P(x_2|B_1)P(B_1) + P(x_2|B_2)P(B_2) = \frac{2}{11}(0.3) + \frac{4}{21}(0.7) = 0.187$$

$$P(x_3) = P(x_3|B_1)P(B_1) + P(x_3|B_2)P(B_2) = \frac{8}{11}(0.3) + \frac{12}{21}(0.7) = 0.618.$$

$$F_X(x) = 0.193\,u(x-1) + 0.187\,u(x-2) + 0.618\,u(x-3).$$

$$f_X(x) = 0.193\,\delta(x-1) + 0.187\,\delta(x-2) + 0.618\,\delta(x-4).$$

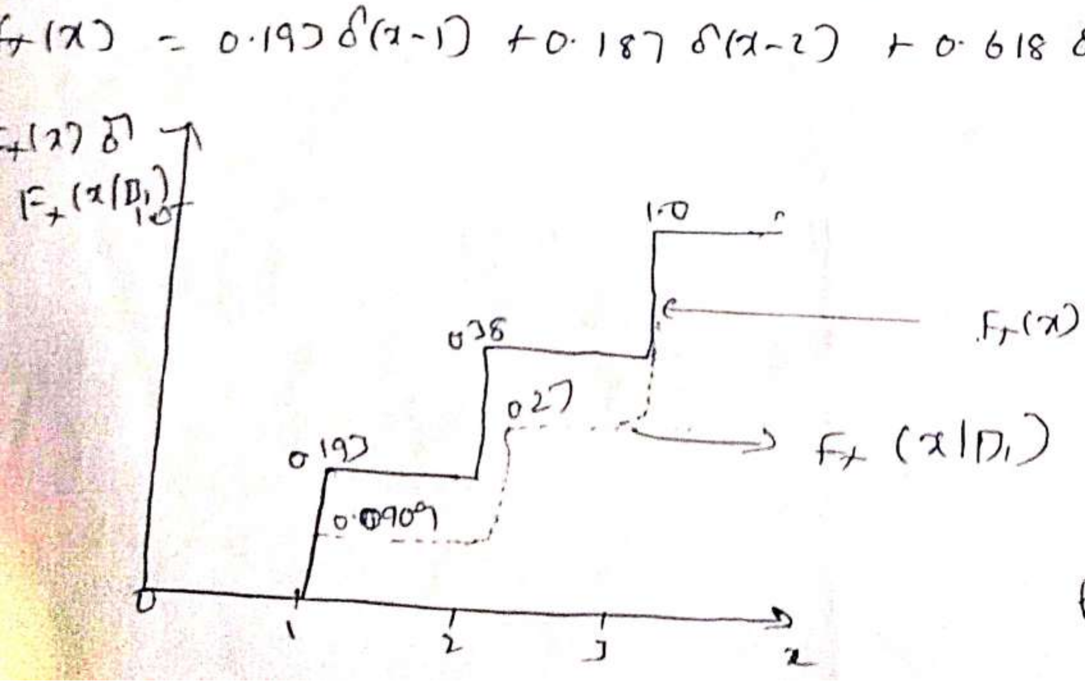


fig (c)

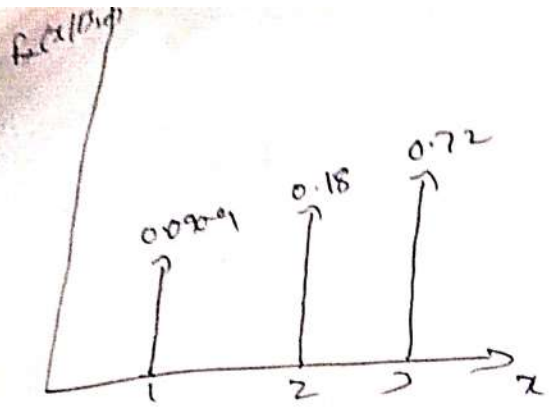


fig (b)

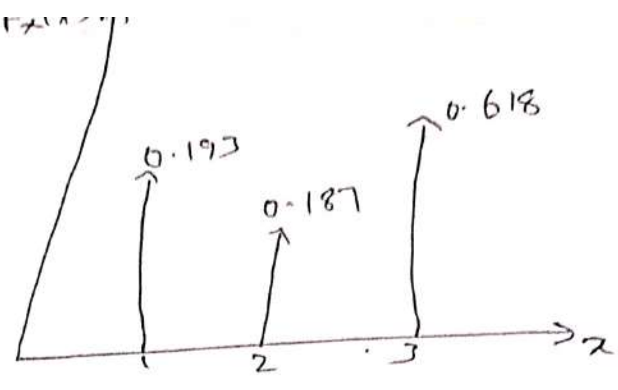


fig (c)

Distributions (a) and densities (b) and (c).

→. Life time IC chips manufactured by a semiconductor manufacture is approximately normally distributed with mean 5×10^6 hours and standard deviation 5×10^5 hours. A main frame manufacturer requires that atleast 95% of a batch should have a lifetime greater than 4×10^6 hours will the deal be made.

sol:- Given $a_x = 5\times10^6$ hours, $\sigma_x = 5\times10^5$ which is a gaussian distributed function

The prob. of getting lifetime greater than 4×10^6 hours is given by $= P\{X > 4\times10^6\} = 1 - P\{X < 4\times10^6\} = 1 - F_X(4\times10^6)$.

$$F_X(4\times10^6) = F\left(\frac{x-a_x}{\sigma_x}\right) = F\left[\frac{4\times10^6 - 5\times10^6}{5\times10^5}\right] = F\left(-\frac{10}{5}\right)$$

$$= F(-2).$$