UNIT - 5

Random processes - spectral characteristics

Introduction:- Let us consider a random signal $x(t)$ then the Fourier transform is given by

$$F.T\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad -(1)$$

Here $X(\omega)$ is refer to as spectrum of $x(t)$ (or) voltage density spectrum and is expressed in volts/Hertz.

If we know the $X(\omega)$ then $x(t)$ can be easily recovered by using inverse Fourier transform ie

$$F^{-1}[X(\omega)] = x(t) = \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \rightarrow (2)$$

Let us consider a random process $X(t)$ and one of its sample function $x(t)$. Let $x_T(t)$ defined the a portion of $x(t)$ over an $(-T, T)$ ie

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{else} \end{cases} \quad -(1)$$

if $x_T(t)$ is bounded ie $\int_{-T}^{T} |x_T(t)| dt < \infty \quad -(2)$

Apply Fourier transform, which we denote $X_T(\omega)$ is given by

$$F[x_T(t)] = X_T(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega t} dt \quad -(3)$$

The energy contained in $x_T(t)$ is given

$$E[T] = \int_{-T}^{T} x_T^2(t) dt = \int_{-T}^{T} x^2(t) dt \quad -(4)$$

Apply parsevall's theorem to eq (4)

$$E[T] = \int_{-T}^{T} x^2(t) dt = \frac{1}{2\pi} \int_{-T}^{T} |X_T(\omega)|^2 d\omega \quad -(5)$$

The eq (5) is divided by $2T$ ~~then~~, we obtain the avg power $P(T)$ in $x(t)$ in $(-T, T)$

$$P(T) = \frac{1}{2T} \int_{-T}^{T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega \quad -(6)$$

Here $\dfrac{|X_T(\omega)|^2}{2T}$ is referred to as power spectral density of $x(t)$

In the above the power is available in the interval $(-T, T)$ only. To get the power in $x(t)$ obtain by taking limit $T \to \infty$.

To find the average power of the R.P $x(t)$ is obtained as

$$P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\{|X_T(\omega)|^2\}}{2T} d\omega \quad - (5)$$

Here $\lim_{T\to\infty} \frac{E\{|X_T(\omega)|^2\}}{2T}$ is refered to as power spectral density ie

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{E\{|X_T(\omega)|^2\}}{2T} \quad - (6)$$

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega \quad - (7)$$

The avg power of the random process is obtained from time domain is mean squared value of time average.

ie $P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)]$ ie

$= A\{E[x^2(t)]\}$.

NOTE: The power spectral density (or) power density spectrum (PSD) is given by $S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[(X_T(\omega))^2]}{2T}$

→ The avg power of the r.p $x(t)$ is given by

$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega).d\omega$$

In time domain $P_{xx} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[x^2(t)]$

→ The power spectral density is FT of $R_{xx}(\tau)$.

$$S_{xx}(\omega) = FT\{R_{xx}(\tau)\}$$
$$= \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau$$

Properties:

→ $S_{xx}(\omega)$ is always non-negative ie $S_{xx}(\omega) \geq 0$

Proof: we know that $S_{xx}(\omega) = \lim_{T\to\infty} E\frac{\{(X_T(\omega))^2\}}{2T}$

The expected value of non-negative function ie $E\{|X_T(\omega)|^2\}$ is always non-negative hence $S_{xx}(\omega) \geq 0$. So ~~S_{xx}~~

(4) → S_{xx} is always real function

w.k.T $S_{xx}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$

since the function $|X_T(\omega)|^2$ is real function hence $S_{xx}(\omega)$ is always real.

(5) → $S_{xx}(-\omega) = S_{xx}(\omega)$ if $x(t)$ real

w.k.T $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{j\omega\tau} d\tau$

put $\tau = -\tau$

$S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(-\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$= S_{xx}(\omega)$

→ The power spectral density at zero freq equals to the area under the curve of auto correlation ie

$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau$

Proof: w.k.T $S_{xx}(\omega) = F\{R_{xx}(\tau)\} = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$\Rightarrow S_{xx}(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot 1 \, d\tau$

$S_{xx}(0) = \int_{-\infty}^{\infty} R_{xx}(\tau) d\tau$

→ The time avg of the mean squared value equals to the area under the power density spectrum $R_{xx}(0)$.

ie $A\{E[x^2(t)]\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = R_{xx}(0)$.

→ If $S_{xx}(\omega)$ is a power density spectral of $x(t)$ then the power density spectrum of derivative of $x(t)$ equals to the ω^2 times $S_{xx}(\omega)$.

ie $S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)$.

→ The Power spectral density and time averaged auto correlation functions are F.T pair

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} A\{R_{xx}(t, t+\tau)\} e^{-j\omega\tau} d\tau$$ and

$$A\{R_{xx}(t, t+\tau)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega$$

→ Relation ship between Power spectrum and autocorrelation function

Statement :- The inverse Fourier transform of Power density spectrum equals to the time average auto correlation function ie

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = A\{R_{xx}(t, t+\tau)\} \quad -①$$

Proof :- w.k.T $S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$

$$\Rightarrow S_{xx}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega) \cdot X_T(\omega)]}{2T} \quad -①$$

w.k.T $$X_T(\omega) = \int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1 \quad -②$$

$$X_T^*(\omega) = \int_{-T}^{T} x(t) e^{j\omega t} dt \quad -③$$

substitute equations ② & ③ in ①

$$\Rightarrow S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t) e^{j\omega t} dt \cdot \int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1\right]$$

$$\Rightarrow S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E[x(t)\, x(t_1)]\, e^{-j\omega(t_1-t)} dt\, dt_1$$

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, e^{-j\omega(t_1-t)} dt\, dt_1 \quad -④$$

Apply inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, e^{-j\omega(t_1-t)} dt\, dt_1\right] e^{j\omega\tau} d\omega$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1, t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\, e^{-j\omega(t_1 - t)} d\omega\right] dt_1\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1 t_1)\left[\frac{1}{2\pi}\int_{-\infty} e^{j\omega(\tau - t_1 + t)} d\omega\right] dt_1\, dt \quad -(5)$$

w.k.T $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\, d\omega = \delta(\tau)$ and also impulse function satisfies symmetry property hence $\delta(\tau - t_1 + t) = \delta(t_1 - t - \tau)$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1 t_1)\left[\delta(t_1 - t - \tau)\right] dt_1\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty} S_{xx}(\omega)\, e^{j\omega\tau} d\omega = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\int_{-T}^{T} R_{xx}(t_1 t_1)\,\delta(t_1 - t - \tau)\, dt_1\right] dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{xx}(t, t+\tau)\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = A\{R_{xx}(t, t+\tau)\}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau} d\omega = A[R_{xx}(t, t+\tau)] \quad - \text{I}$$

From equation (I) the inverse FT of Power density spectrum is equals to the time average auto correlation function. Now from direct fourier transform

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} A[R_{xx}(t, t+\tau)]\, e^{-j\omega\tau}\, d\tau \quad - \text{II}$$

From equation II Power density spectrum is a Fourier transform of time average auto correlation function.

case ii: consider $x(t)$ is atleast w.s.s then

$$A\{R_{xx}(t, t+\tau)\} = R_{xx}(\tau)$$

Then from II $\quad S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau \quad - \text{III}$

similarly from eq I

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega \quad - \text{IV}$$

Power density spectrum is F.T of auto correlation and auto correlation is inverse Fourier of power density spectrum.
Hence power spectral density and auto correlation are F.T pair

i.e $R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$

$\therefore$ equations III & IV are also called as wiener - kimtchien relations

$\Rightarrow$ Determine which of the following functions are valid power spectral density (i) $\frac{\omega^2}{\omega^6+3\omega^2+3}$ (ii) $\exp[-(\omega-1)^2]$ (iii) $\frac{\omega^2}{\omega^4+1}-\delta(\omega)$ (iv) $\frac{\omega^4}{1+\omega^2+i\omega^6}$

sol (i) Let $S_{XX}(\omega) = \frac{\omega^2}{\omega^6+3\omega^2+3}$

consider $S_{XX}(-\omega) = \frac{(-\omega)^2}{(-\omega)^6+3(-\omega)^2+3} = \frac{\omega^2}{\omega^6+3\omega^2+3} = S_{XX}(\omega)$

$S_{XX}(-\omega) = S_{XX}(\omega)$ Hence given function is valid psd

(ii) $\exp[-(\omega-1)^2]$

let $S_{XX}(\omega) = e^{-(\omega-1)^2}$

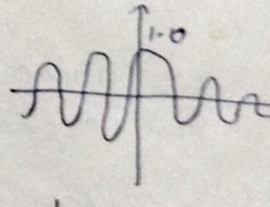
consider $S_{XX}(-\omega) = e^{-(-\omega-1)^2} \neq S_{XX}(\omega)$ not valid psd

(iii) let $S_{XX}(\omega) = \frac{\omega^2}{\omega^4+1} - \delta(\omega)$

consider $S_{XX}(-\omega) = \frac{\omega^2}{\omega^4+1} - \delta(-\omega) = \frac{\omega^2}{\omega^4+1} - \delta(\omega)$ It satisfies psd

(iv) $S_{XX}(-\omega) = \frac{\omega^4}{1+\omega^2+i\omega^6} = \frac{(-\omega)^2}{1+(-\omega)^2+i(-\omega)^6} = S_{XX}(\omega)$ It satisfies

$\Rightarrow$ The PSD of R.P is given by $S_{XX}(\omega) = \begin{cases} \pi & |\omega| < 1 \\ 0 & \text{elsewhere} \end{cases}$



Find its auto correlation function

sol w.k.T $R_{XX}(\tau) = F^{-1}(S_{XX}(\omega))$

$= \int_{-\infty}^{\infty} \frac{1}{2\pi} S_{XX}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-1}^{1} \pi e^{j\omega\tau} d\omega$

$= \frac{1}{2}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1} = \frac{1}{2j\tau}\left[e^{j\tau} - e^{-j\tau}\right] = \frac{1}{\tau}\left[\frac{e^{j\tau}-e^{-j\tau}}{2j}\right]$

$= \frac{\sin\tau}{\tau} = Sa(\tau)$

→ For a R.P $x(t)$ assume that $R_{xx}(\tau)$ equals to $R_{xx}(\tau) = P\exp\left(\frac{-\tau^2}{2a^2}\right)$ where $P>0$ and $a>0$ are constants find the PSD of $x(t)$?

sol:
$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} P\exp\left(\frac{-\tau^2}{2a^2}\right) e^{-j\omega\tau} d\tau$$

$$= P\int_{-\infty}^{\infty} \exp\left(-\frac{\tau^2}{2a^2} - j\omega\tau\right) d\tau$$

w.k.t
$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx} dx = \frac{\sqrt{\pi}}{a} e^{\frac{b^2}{4a^2}} \quad a>0$$

here $a^2 = \frac{1}{2a^2}$ and $b = -j\omega$

$$= P \cdot \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}a} e^{(-j\omega)^2 / 4\left(\frac{1}{2a^2}\right)} = P \cdot \sqrt{2\pi} \cdot a \, e^{-\omega^2/2/a^2}$$

$$= \sqrt{2\pi} \cdot Pa \cdot e^{-\omega^2 a^2/2}$$

→ Here the auto correlation function of WSS RP is $R_{xx}(\tau) = a\exp\left(-\left(\frac{\tau}{b}\right)^2\right)$ and find PSD and also normalised avg power density

sol:- $R_{xx}(\tau) = a\exp\left(-\left(\frac{\tau}{b}\right)^2\right)$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} a e^{-(\tau/b)^2} e^{-j\omega\tau} d\tau$$

$$= a\int_{-\infty}^{\infty} e^{-\tau^2/b^2 - j\omega\tau} d\tau = a\int_{-\infty}^{\infty} e^{-\left(\frac{1}{b^2}\right)\tau^2 + (-j\omega)\tau} d\tau$$

$$= a \cdot \frac{\sqrt{\pi}}{\frac{1}{b}} e^{-\omega^2 b^2/4}$$

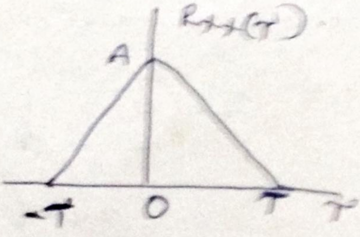
$$= ab\sqrt{\pi}\, e^{-\omega^2 b^2/4}$$

Time avg power density
$$P_{xx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{\pi}ab\, e^{-\omega^2 b^2/4} d\omega$$

$$= \frac{ab}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\omega^2 b^2/4} d\omega \qquad \left[\because \int_{-\infty}^{\infty} e^{-x^2(a)} dx = \sqrt{\frac{\pi}{a}}\right.$$

$$= \frac{ab}{2\sqrt{\pi}} \sqrt{\frac{\pi}{b^2} \cdot 4} = a$$

→ Find the power spectrum for the random process $x(t)$ with auto correlation function is shown in fig



sol: when $-T < \tau < 0$

$(-T, 0)$ $\quad (0, A)$

$x_1 \; y_1 \quad x_2 \; y_2$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$R_{xx}(\tau) = \frac{A}{T}(\tau + T) = A\left(\frac{\tau}{T} + 1\right)$$

when $0 < \tau < T$; $(0, A)$ $\quad (T, 0)$

$x_1 \; y_1 \quad x_2 \; y_2$

$$R_{xx}(\tau) - A = -\frac{A}{T}(\tau - 0)$$

$$R_{xx}(\tau) = A - \frac{A}{T}(\tau) = A - \left(1 - \frac{\tau}{T}\right)$$

$$\therefore \; R_{xx}(\tau) = \begin{cases} A\left(1 - \frac{|\tau|}{T}\right) & -T < \tau < T \\ 0 & \text{else} \end{cases}$$

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-T}^{T} A\left(1 - \frac{|\tau|}{T}\right) e^{-j\omega\tau}\, d\tau$$

$$= \int_{-T}^{0} A\left(1 + \frac{\tau}{T}\right) e^{-j\omega\tau}\, d\tau + A \int_{0}^{T} 1 - \frac{\tau}{T}\, e^{-j\omega\tau}\, d\tau$$

$$= A - \int_{-T}^{0} e^{-j\omega\tau} + \frac{\tau}{T} e^{-j\omega\tau}\, d\tau + A \int_{0}^{T} e^{-j\omega\tau} - \frac{\tau}{T} e^{-j\omega\tau}\, d\tau$$

consider $\displaystyle\int_{-T}^{0} \frac{\tau}{T} e^{-j\omega\tau}\, d\tau = \frac{1}{T} \int_{-T}^{0} \tau e^{-j\omega\tau}\, d\tau$

$$= \frac{1}{T}\left(\tau \cdot \frac{e^{-j\omega\tau}}{-j\omega} + \int \frac{e^{-j\omega\tau}}{j\omega}\, d\tau \right) = \frac{1}{T}\left[\tau \frac{e^{-j\omega\tau}}{-j\omega} + \frac{e^{-j\omega\tau}}{\omega^2} \right]\Bigg|_{-T}^{0}$$

$$= A \cdot \frac{e^{-j\omega\tau}}{-j\omega}\Bigg|_{-T}^{0} + \frac{A}{T}\left[\frac{\tau e^{-j\omega\tau}}{j\omega} + \frac{e^{-j\omega\tau}}{\omega^2} \right]\Bigg|_{-T}^{0}$$

$$= \frac{-A}{j\omega} + \frac{A}{j\omega} e^{j\omega T} + \frac{A}{T}\left[0 + \frac{1}{\omega^2} - T\frac{e^{j\omega T}}{j\omega} - \frac{1}{\omega^2} e^{j\omega T} \right]$$

$$= \frac{A}{j\omega}\left(e^{j\omega T} - 1\right) + \frac{A}{T}\left[\frac{1}{\omega^2} - T\frac{e^{j\omega T}}{j\omega} - \frac{e^{j\omega T}}{\omega^2} \right]$$

$$= \frac{A\cancel{e^{j\omega T}}}{\cancel{j\omega}} - \frac{A}{j\omega} + \frac{A}{\omega^2 T} - \frac{\cancel{Ae^{j\omega T}}}{j\omega} - \frac{Ae^{j\omega T}}{\omega^2 T}$$

$$= -\frac{Ae^{j\omega T}}{\omega^2 T} - \frac{A}{j\omega} + \frac{A}{\omega^2 T}$$

→ consider $A\int_0^T \left(e^{-j\omega\tau} - \frac{\tau}{T} e^{-j\omega\tau}\right) d\tau$

$$= A \cdot \frac{e^{-j\omega\tau}}{-j\omega}\Big|_0^T - \frac{A}{T}\left\{\tau \frac{e^{-j\omega\tau}}{-j\omega} + \int \frac{e^{-j\omega\tau}}{j\omega} d\tau\right\}$$

$$= A\left[-\frac{e^{-j\omega T}}{j\omega} + \frac{1}{j\omega}\right] - \frac{A}{T}\left\{-\frac{\tau e^{-j\omega\tau}}{j\omega} + \frac{e^{-j\omega\tau}}{\omega^2}\right\}\Big|_0^T$$

$$= \frac{-Ae^{j\omega T}}{j\omega} + \frac{A}{j\omega} - \frac{A}{T}\left\{-\frac{T e^{-j\omega T}}{j\omega} + \frac{e^{-j\omega T}}{\omega^2} - \frac{1}{\omega^2}\right\}$$

$$= -\frac{\cancel{Ae^{j\omega T}}}{j\omega} + \frac{A}{j\omega} + \frac{\cancel{Ae^{-j\omega T}}}{j\omega} - \frac{A}{T}\frac{e^{-j\omega T}}{\omega^2} + \frac{A}{T\omega^2}$$

$$= \frac{A}{j\omega} - \frac{A}{T}\frac{e^{-j\omega T}}{\omega^2} + \frac{A}{T\omega^2}$$

$$\therefore S_{xx}(\omega) = \frac{A}{\omega^2 T} - \cancel{\frac{A}{j\omega}} - \frac{Ae^{j\omega T}}{\omega^2 T} + \cancel{\frac{A}{j\omega}} - \frac{A}{T}\frac{e^{-j\omega T}}{\omega^2} + \frac{A}{T\omega^2}$$

$$= \frac{2A}{\omega^2 T} - \frac{A}{T}\frac{e^{-j\omega T}}{\omega^2} - \frac{Ae^{j\omega T}}{\omega^2 T}$$

$$= \frac{2A}{\omega^2 T} - \frac{2A}{\omega^2 T}\left[\frac{e^{j\omega T} + e^{-j\omega T}}{2}\right] = \frac{2A}{\omega^2 T} - \frac{2A}{\omega^2 T}\cos\omega T = \frac{2A}{\omega^2 T}\{1 - \cos\omega T\}$$

$$= \frac{2A}{\omega^2 T} 2\sin^2\frac{\omega T}{2} = \frac{4A}{\omega^2 T}\frac{\sin^2\frac{\omega T}{2}}{(\omega T/2)^2}\left(\frac{\omega T}{2}\right)^2$$

$$= \frac{4A}{\omega^2 T}\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\omega T/2}\right)^2 \frac{\omega^2 T^2}{4} = AT\, Sa^2\frac{\omega T}{2}$$

→ Find the power spectral density for $R_{xx}(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau$

sol: $R_{xx}(\tau) = \frac{A_0^2}{2}\cos\omega_0\tau$

$$S_{xx}(\omega) = F\{R_{xx}(\tau)\} = F\left\{\frac{A_0^2}{2}\cos\omega_0\tau\right\} = \frac{A_0^2}{2} F\left\{\frac{e^{j\omega_0\tau} + e^{-j\omega_0\tau}}{2}\right\}$$

$$= \frac{A_0^2}{4}\left[F\{e^{j\omega_0\tau}\} + F\{e^{-j\omega_0\tau}\}\right]$$

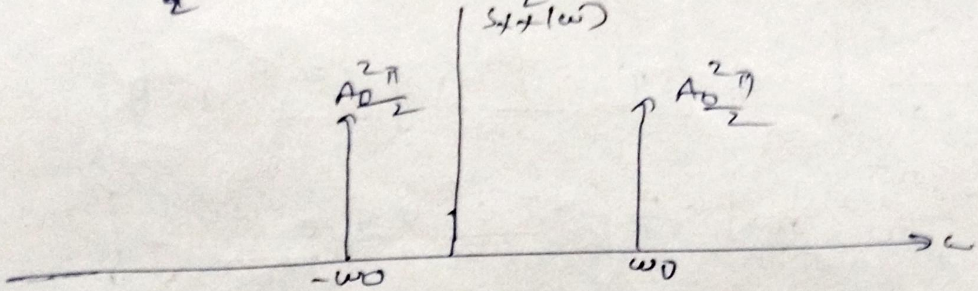
Fourier Transform of $e^{j\omega_0\tau} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

$e^{-j\omega_0\tau} \longleftrightarrow 2\pi\delta(\omega+\omega_0)$

$1 \longleftrightarrow 2\pi\delta(\omega)$.

$$\therefore S_{xx}(\omega) = \frac{A_0^2}{4}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$= \frac{A_0^2}{2}\pi\,\delta(\omega-\omega_0) + \frac{A_0^2}{2}\pi\,\delta(\omega+\omega_0).$$



Power spectrum

$\Rightarrow$ If the auto correlation of a w.s.s is given by $R_{xx}(\tau) = k e^{-k|\tau|}$ show that the spectral density is given by $S_{xx}(\omega) = \dfrac{2}{1+\left(\frac{\omega}{k}\right)^2}$

sol: $S_{xx}(\omega) = F\{R_{xx}(\tau)\} = \int_{-\infty}^{\infty} k e^{-k|\tau|} e^{-j\omega\tau}\,d\tau$

$$= k\left[\int_{-\infty}^{0} e^{-k(-\tau)-j\omega\tau}\,d\tau + \int_{0}^{\infty} e^{-k\tau-j\omega\tau}\,d\tau\right]$$

$$= k\left[\int_{-\infty}^{0} e^{k\tau-j\omega\tau}\,d\tau + \int_{0}^{\infty} e^{-k\tau-j\omega\tau}\,d\tau\right]$$

$$= k\left[\int_{-\infty}^{0} e^{(k-j\omega)\tau}\,d\tau + \int_{0}^{\infty} e^{-(k+j\omega)\tau}\,d\tau\right]$$

$$= k\left[\frac{e^{(k-j\omega)\tau}}{k-j\omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(k+j\omega)\tau}}{-(k+j\omega)}\Bigg|_{0}^{\infty}\right]$$

$$= k\left[\frac{1}{k-j\omega}\right] + \frac{k}{k+j\omega}$$

$$= \frac{k(k+j\omega)+k(k-j\omega)}{(k-j\omega)(k+j\omega)} = \frac{2k^2}{k^2+\omega^2} = \frac{2\cancel{k^2}}{\cancel{k^2}\left(1+\left(\frac{\omega}{k}\right)^2\right)}$$

$$S_{xx}(\omega) = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

⇒ The PSd of $x(t)$ is given by

$$S_{xx}(\omega) = \begin{cases} 1+\omega^2 & \text{for } |\omega| < 1 \\ 0 & \text{else} \end{cases}$$ Find out the autocorrelation function

Sol: The auto correlation function is

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1} (1+\omega^2)\, e^{j\omega\tau}\, d\omega$$

$$R_{xx}(\tau) = \frac{1}{2\pi}\left[\int_{-1}^{1} e^{j\omega\tau}\, d\omega + \int_{-1}^{1} \omega^2 e^{j\omega\tau}\, d\omega\right]$$

Now take $\int_{-1}^{1} \omega^2 e^{j\omega\tau}\, d\omega = \frac{e^{j\omega\tau}}{j\tau}\cdot\omega^2\Big|_{-1}^{1} - \int_{-1}^{1} \frac{e^{j\omega\tau}}{j\tau}(2\omega)\, d\omega$

$$= \frac{e^{j\tau} - e^{-j\tau}}{j\tau} - \left[\frac{2e^{j\omega\tau}}{(j\tau)^2}\cdot\omega\Big|_{-1}^{1} - \int_{-1}^{1} \frac{2e^{j\omega\tau}}{(j\tau)^2}\, d\omega\right]$$

$$= \frac{e^{j\tau} - e^{-j\tau}}{j\tau} - \frac{2(e^{j\tau} + e^{-j\tau})}{-\tau^2} + \frac{2}{-\tau^2}\,\frac{e^{j\omega\tau}}{j\tau}\Big|_{-1}^{1}$$

$$= \frac{e^{j\tau} - e^{-j\tau}}{j\tau} + \frac{2(e^{j\tau} + e^{-j\tau})}{\tau^2} - \frac{2(e^{j\tau} - e^{-j\tau})}{j\tau^3}$$

Also $\int_{-1}^{1} e^{j\omega\tau}\, d\omega = \frac{e^{j\tau} - e^{-j\tau}}{j\tau}$

$$\therefore R_{xx}(\tau) = \frac{1}{2\pi}\left[\frac{e^{j\tau} - e^{-j\tau}}{j\tau} + \frac{e^{j\tau} - e^{-j\tau}}{j\tau} + \frac{2}{\tau^2}(e^{j\tau} + e^{-j\tau}) - \frac{2}{j\tau^3}(e^{j\tau} - e^{-j\tau})\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\sin\tau}{\tau} + \frac{4\cos\tau}{\tau^2} - \frac{4}{\tau^3}\sin\tau\right]$$

$$R_{xx}(\tau) = \frac{1}{\pi}\left[\frac{2\sin\tau}{\tau} + \frac{2\cos\tau}{\tau^2} - \frac{2}{\tau^3}\sin\tau\right]$$

$$R_{xx}(\tau) = \frac{2}{\pi\tau^3}\left[\tau^2\sin\tau + \tau\cos\tau - \sin\tau\right]$$

⇒ Consider the random process $x(t) = A\cos(\omega t + \theta)$ where A and ω are real constants and θ is a R.V uniformly distributed over $(0, 2\pi)$. Find the average power P_{xx}.

Sol: we know that $P_{xx} = A[E[x^2(t)]]$

now $x(t) = A\cos(\omega t+\theta)$ and $f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{else} \end{cases}$

$$E[x^2(t)] = \int_{-\infty}^{\infty} x^2(t) f_\theta(\theta)\, d\theta = \int_0^{2\pi} A^2\cos^2(\omega t+\theta) \cdot \frac{1}{2\pi} \cdot d\theta$$

$$= \frac{A^2}{2\pi}\left[\int_0^{2\pi} \frac{1+\cos(2\omega t+2\theta)}{2}\, d\theta\right] = \frac{A^2}{2\pi}\left[\int_0^{2\pi} d\theta + \int_0^{2\pi}\cos(2\omega t+2\theta)\,d\theta\right]$$

$$= \frac{A^2}{4\pi}\left[2\pi + (-1)\frac{\sin(2\omega t+2\theta)}{2}\Big|_0^{2\pi}\right] = \frac{A^2}{4\pi}\left[2\pi - 4\sin(2\omega t)\right]$$

$$E[x^2(t)] = \frac{A^2}{2} - \frac{A^2}{\pi}\sin\omega t$$

The time avg power is $P_{xx} = A[E[x^2(t)]]$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\left[\frac{A^2}{2} - \frac{A^2}{\pi}\sin(2\omega t)\right]dt$$

$$= \frac{1}{2T} \cdot \frac{A^2}{2}(2T) - 0 = \frac{A^2}{2}.$$

V. Imp

→ A random process has PSD $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^4}$ find the average power

Given $S_{xx}(\omega) = \frac{6\omega^2}{1+\omega^4}$

$$P_{xx} = R_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{6\omega^2}{1+\omega^4}\,d\omega$$

$$P_{xx} = \frac{6}{2\pi}\int_{-\infty}^{\infty}\frac{\omega^2}{1+\omega^4}\,d\omega \qquad \left[\because \int_{-\infty}^{\infty}\frac{x^2}{a+x^4}\,dx = \frac{\pi}{2\sqrt{2}\cdot a}\right]$$

Here $a=1$

$$= \frac{6}{2\pi} \cdot \frac{\pi}{2\sqrt{2}}$$

$$P_{xx} = \frac{3}{2\sqrt{2}} = 1.06 \text{ watts}$$

Cross Power density spectrum

consider two random processes $x(t)$ and $y(t)$ and one of their sample functions $x(t)$ and $y(t)$ respectively. Let $x_T(t)$ and $y_T(t)$ are the portions of two sample functions $x(t)$ and $y(t)$ over an interval $(-T,T)$.

ie $x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{else} \end{cases}$ — ①

$y_T(t) = \begin{cases} y(t) & -T < t < T \\ 0 & \text{else} \end{cases}$ — ②

Apply Fourier transform

$$X_T(\omega) = \int_{-T}^{T} x_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} x(t) e^{-j\omega t}\, dt \quad -③$$

$$Y_T(\omega) = \int_{-T}^{T} y_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} y(t) e^{-j\omega t}\, dt \quad -④$$

The cross Power of two Processes over an $(-T,T)$ is given by

$$P_{xy}(T) = \frac{1}{2T} \int_{-T}^{T} x_T(t)\, y_T(t)\, dt$$

$$= \frac{1}{2T} \int_{-T}^{T} x(t) y(t) dt$$

Apply Parseval's theorem $\int_{-T}^{T} x^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

$$\Rightarrow P_{xy}(T) = \frac{1}{2T} \int_{-T}^{T} x(t) y(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{X_T^*(\omega) Y_T(\omega)}{2T} d\omega \quad -⑤$$

Average Cross Power is obtained by taking expected value of $P_{xy}(T)$.

$$\Rightarrow \overline{P_{xy}(T)} = \frac{1}{2T} \int_{-T}^{T} E[x(t) y(t)] dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T} d\omega$$

$$\Rightarrow \overline{P_{xy}(T)} = \frac{1}{2T} \int_{-T}^{T} R_{xy}(t,t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T} d\omega \quad -⑥$$

∴ The total average cross power P_{xy} is obtained by taking $T \to \infty$

$$\Rightarrow P_{xy} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R_{xy}(t,t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{E[X_T^*(\omega) Y_T(\omega)]}{2T} d\omega$$

— ⑦

Here the term $\lim_{T\to\infty} \frac{E\left[X_T^*(\omega) Y_T(\omega)\right]}{2T}$ is called cross Power density spectrum. and it is denoted as $S_{XY}(\omega)$.

$$\therefore S_{XY}(\omega) = \lim_{T\to\infty} \frac{E\left[X_T^*(\omega) Y_T(\omega)\right]}{2T} \rightarrow I$$

The total cross Power P_{XY} is given by

$$\boxed{P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, d\omega}$$

From eq ① we note that the average cross Power equals to the time average cross correlation function.

NOTE:- cross Power density spectrum is the F.T of cross correlation fun

$$S_{XY}(\omega) = FT\{R_{XY}(\tau)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau.$$

Properties:-

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega)$.

Proof:- we know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau} d\tau$$

$$S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{j\omega\tau} d\tau$$

put $\tau = -\tau$

$$\Rightarrow S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{YX}(-\tau)\, e^{-j\omega\tau} d\tau$$

we know that $R_{YX}(-\tau) = R_{XY}(\tau)$

$$\Rightarrow S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau$$

$$\boxed{S_{YX}(-\omega) = S_{XY}(\omega)}$$

2. $Re\{S_{XY}(\omega)\}$ and $Re\{S_{YX}(\omega)\}$ are even functions of 'ω'.

Proof: $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)\{\cos\omega\tau - j\sin\omega\tau\}\, d\tau$$

$$\Rightarrow Re\{S_{XY}(\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau\, d\tau$$

$$\Rightarrow Re\{S_{XY}(-\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos(-\omega)\tau\, d\tau$$

$$\Rightarrow Re\{S_{XY}(-\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)\cos\omega\tau\, d\tau = Re\{S_{XY}(\omega)\}$$

similarly $Re\{S_{YX}(\omega)$ is also even function.

3. $Img\{S_{XY}(\omega)\}$ and $Img\{S_{YX}(\omega)\}$ are odd functions of 'ω'.

Proof:-
$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{XY}(\tau)\left[\cos\omega\tau - j\sin\omega\tau\right] d\tau$$

$$Img\{S_{XY}(\omega)\} = \int_{-\infty}^{\infty} R_{XY}(\tau)(-\sin\omega\tau)\, d\tau$$

$$= -\int_{-\infty}^{\infty} R_{XY}(\tau)\sin(\omega\tau)\, d\tau$$

$$Img[S_{XY}(-\omega)] = -\int_{-\infty}^{\infty} R_{XY}(\tau)\sin(-\omega\tau)\, d\tau$$

$$= -\int_{-\infty}^{\infty} R_{XY}(\tau)(-\sin\omega\tau)\, d\tau = -Img\{S_{XY}(\omega)\}$$

4. $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$ If two processes $X(t)$ and $Y(t)$ are orthogonal

Proof:- If $X(t)$ and $Y(t)$ are said to be orthogonal if its cross correlation functions are zero ie $R_{XY}(\tau) = 0$, $R_{YX}(\tau) = 0$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau = 0$$

similarly
$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau} d\tau = 0$$

5. If $X(t)$ and $Y(t)$ are uncorrelated and have constant mean values $\bar{X}$ and $\bar{Y}$ then $S_{XY}(\omega) = S_{YX}(\omega)$ is equals to

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi\bar{X}\bar{Y}\,\delta(\omega).$$

Proof:-
$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} E[X(t)\,Y(t+\tau)]\, e^{-j\omega\tau} d\tau$$

If $x(t)$ and $y(t)$ are uncorrelated and have constant mean $\bar{x}, \bar{y}$

then $E[x(t)\, y(t+\tau)] = E[x(t)]\, E[y(t+\tau)]$

$= \bar{x}\bar{y}$

$\therefore S_{xy}(\omega) = \int_{-\infty}^{\infty} \bar{x}\bar{y}\, e^{-j\omega\tau}\, d\tau = \bar{x}\bar{y} \int_{-\infty}^{\infty} (1)\, e^{-j\omega\tau}\, d\tau$

$= \bar{x}\bar{y}\, 2\pi\, \delta(\omega)$.

$\therefore S_{xy}(\omega) = 2\pi\, \bar{x}\cdot\bar{y}\, \delta(\omega)$

6. Cross power density spectrum and time average of cross correlation function are fourier transform pair.

$A[R_{xy}(t, t+\tau)] \xleftrightarrow{F} S_{xy}(\omega)$.

similarly $A[R_{yx}(t, t+\tau)] \xleftrightarrow{F} S_{yx}(\omega)$.

<u>Relationship between cross power spectrum and cross correlation function</u>

<u>Statement</u>:- The inverse Fourier transform of cross power spectrum equals to the time average of cross correlation function ie

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega = A[R_{xy}(t, t+\tau)].$$

<u>Proof</u>:- Consider $x(t)$ and $y(t)$ are two real random processes.

Let $x_T(t)$ and $y_T(t)$ are defined over an $(-T, T)$.

The fourier transforms are

$$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{-j\omega t}\, dt \quad -(1)$$

$$Y_T(\omega) = \int_{-T}^{T} y(t)\, e^{-j\omega t_1}\, dt_1 \quad -(2)$$

we know that, cross power spectrum

$$S_{xy}(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}$$

$$\Rightarrow S_{xy}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x(t)\, e^{j\omega t}\, dt \cdot \int_{-T}^{T} y(t)\, e^{-j\omega t_1}\, dt_1\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E[x(t)\, y(t)]\, e^{-j\omega(t_1 - t)}\, dt\, dt_1 \quad \rightarrow (3)$$

Take inverse Fourier Transform of eq (3) on both sides,

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t)\,Y(t_1)\}\, e^{-j\omega(t_1-t)}\, dt\, dt_1\right) e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\,d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t,t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t_1-t)}\, e^{j\omega\tau}\,d\omega\; dt_1\, dt\right).$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t,t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(\tau-t_1+t)}\, d\omega\right\} dt_1\, dt$$

We know that $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}\,d\omega = \delta(\tau)$ and impulse satisfies even symmetry property hence

$$\delta(\tau-t_1+t) = \delta(t_1-t-\tau).$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\,d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_{-T}^{T} R_{XY}(t,t_1)\,\delta(t_1-t-\tau)\, dt_1\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\,d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t+\tau)\,dt$$

$$= A\left[R_{XY}(t,t+\tau)\right]$$

$$\boxed{\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\,d\omega = A\left[R_{XY}(t,t+\tau)\right]} \quad \text{—} (4)$$

Equation (4) represents the inverse Fourier transform of cross power spectrum equals to the time average of cross correlation function.

Now by direct transform

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} A\left(R_{XY}(t,t+\tau)\right)e^{-j\omega\tau}\,d\tau \quad \text{—} (5)$$

From equations (4) & (5) cross power spectrum and cross correlation functions are Fourier Transform pair.

If $X(t)$ and $Y(t)$ are jointly wide sense stationary

Then $A\left[R_{XY}(t,t+\tau)\right] = R_{XY}(\tau)$

$$\Rightarrow S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau \quad - (6)$$

$$\Rightarrow R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega\tau} d\omega \quad - (7)$$

From equations (6) & (7) cross power spectrum and cross correlation functions are Fourier transform pair.

→ Find the cross power spectrum of cross correlation function $R_{xy}(\tau) = \frac{A^2}{2} \sin\omega_0\tau$.

Sol: Given $R_{xy}(\tau) = \frac{A^2}{2}\sin\omega_0\tau$

w.k.t $S_{xy}(\omega) = F.T\{R_{xy}(\tau)\} = F.T\{\frac{A^2}{2}\sin\omega_0\tau\}$

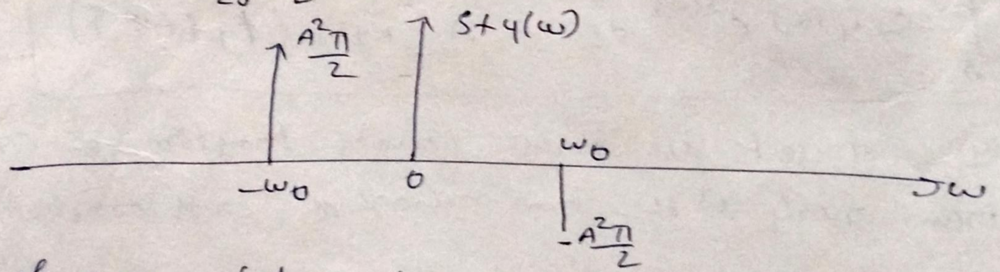
$$= \frac{A^2}{2} F.T\left\{\frac{e^{j\omega_0\tau} - e^{-j\omega_0\tau}}{2j}\right\}$$

$$= \frac{A^2}{4j}\left[F\{e^{j\omega_0\tau}\} - F\{e^{-j\omega_0\tau}\}\right]$$

$e^{j\omega_0\tau} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$, $e^{-j\omega_0\tau} \leftrightarrow 2\pi\delta(\omega+\omega_0)$.

$$S_{xy}(\omega) = \frac{A^2}{4j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$S_{xy}(\omega) = \frac{A^2\pi}{2j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$



F.T of some useful functions

$e^{-j\omega_0\tau} \xleftrightarrow{F} 2\pi\delta(\omega+\omega_0)$

$e^{j\omega_0\tau} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$

$\cos\omega_0\tau \xleftrightarrow{F} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$\sin\omega_0\tau \xleftrightarrow{F} \pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

$e^{-a\tau}u(\tau) \xleftrightarrow{F} \frac{1}{a+j\omega}$ $a>0$

$\tau e^{-a\tau}u(t) \xleftrightarrow{F} \frac{1}{(a+j\omega)^2}$ $a>0$

$\tau^2 e^{-a\tau}u(\tau) \leftrightarrow \frac{2}{(a+j\omega)^3}$

$\tau^n e^{-a\tau}u(\tau) \leftrightarrow \frac{n!}{(a+j\omega)^{n+1}}$ $a>0$

$e^{-a|\tau|} \leftrightarrow \frac{2a}{a^2+\omega^2}$

$e^{-a|\tau|} * e^{-a|\tau|} \leftrightarrow \left(\frac{2a}{a^2+\omega^2}\right)^2$

→ Find the auto correlation functions for the given Power spectral densities (i) $S_{xx}(\omega) = \frac{3}{2+j\omega}$ (ii) $S_{xx}(\omega) = \frac{4}{(6+j\omega)^2}$

Sol: (i) $S_{xx}(\omega) = \frac{3}{2+j\omega}$

w.k.T $R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)] = F^{-1}\left[\frac{3}{2+j\omega}\right] = 3F^{-1}\left[\frac{1}{2+j\omega}\right]$

w.k.T $e^{-a\tau}u(\tau) \xleftrightarrow{F} \frac{1}{a+j\omega}$

$\therefore R_{xx}(\tau) = 3\cdot e^{-2\tau}u(\tau)$.

(ii) $S_{xx}(\omega) = \frac{4}{(6+j\omega)^2}$

w.k.T $R_{xx}(\tau) = F^{-1}(S_{xx}(\omega))$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{(6+j\omega)^2}e^{j\omega\tau}\,d\omega = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{(6+j\omega)^2}e^{j\omega\tau}\,d\omega$$

$$= 4\left[F^{-1}\left(\frac{1}{(6+j\omega)^2}\right)\right] = 4\cdot\tau e^{-6\tau}u(\tau).$$

(iii) $S_{xx}(\omega) = \frac{10}{9+\omega^2}$

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{10}{9+\omega^2}e^{j\omega\tau}\,d\omega$$

$$= \frac{10}{6}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\times3}{3^2+\omega^2}e^{j\omega\tau}\,d\omega\right]$$

$$= \frac{5}{3}\left[F^{-1}\frac{2\times3}{3^2+\omega^2}\right] = \frac{5}{3}e^{-a|\tau|}$$

→ Find out the cross covelation of psd $S_{xy}(\omega) = \frac{1}{25+\omega^2}$

Sol: $R_{xy}(\tau) \Leftrightarrow F^{-1}[S_{xy}(\omega)] = F^{-1}\left\{\frac{1}{25+\omega^2}\right\}$

w.k.T $\frac{2a}{a^2+\omega^2} \xleftrightarrow{F.T} e^{-|a|\tau|}$ $a>0$.

(or) $\frac{1}{a^2+\omega^2} \Leftrightarrow \frac{1}{2a}e^{-a|\tau|}$

$\therefore \frac{1}{25+\omega^2} \leftrightarrow \frac{1}{2\times5} e^{-5|\tau|}$
$\quad = \frac{1}{10} e^{-5|\tau|}$.

$R_{XY}(\tau) = \frac{1}{10} e^{-5|\tau|}$

→ Given the cross spectral density of two R.P $X(t)$ & $Y(t)$

$$S_{XY}(\omega) = \frac{1}{-\omega^2 + j4\omega + 4} \quad \text{find } R_{XY}(\tau)$$

Sol

$$S_{XY}(\omega) = \frac{1}{(2+j\omega)^2}$$

$$R_{XY}(\tau) = F^{-1}\{S_{XY}(\omega)\}$$

$$= F^{-1}\left\{\frac{1}{(2+j\omega)^2}\right\} = \tau e^{-2\tau} u(t).$$

→ The cross power spectrum of $X(t)$ and $Y(t)$ is defined by

$$S_{XY}(\omega) = \begin{cases} K + \frac{j\omega}{W} & -W < \omega < W \\ 0 & \text{else} \end{cases}$$

find $R_{XY}(\tau)$.

$$R_{XY}(\tau) = \frac{1}{2\pi}\left(\int_{-W}^{W} K e^{j\omega\tau} d\omega + \frac{j}{W}\int_{-W}^{W} \omega e^{j\omega\tau} d\omega\right)$$

$$\int_{-W}^{W} \omega e^{j\omega\tau} d\omega = \frac{e^{j\omega\tau}}{j\tau}(\omega) - \int_{-W}^{W} \frac{e^{j\omega\tau}}{j\tau} d\omega$$

$$= \left(\frac{\omega e^{j\omega\tau}}{j\tau} - \frac{e^{j\omega\tau}}{(j\tau)^2}\right)_{-W}^{W}$$

$$= e^{jW\tau}\left[\frac{W}{j\tau} + \frac{1}{\tau^2}\right] - e^{-jW\tau}\left\{\frac{-W}{j\tau} + \frac{1}{\tau^2}\right)$$

and $\int_{-W}^{W} e^{j\omega\tau} d\omega = \frac{e^{jW\tau} - e^{-jW\tau}}{j\tau}$

$$R_{XY}(\tau) = \frac{K}{2\pi}\left(\frac{e^{jW\tau} - e^{-jW\tau}}{j\tau}\right) + \frac{j}{2\pi W}\left[\frac{W}{j\tau}\left(e^{-jW\tau} + e^{jW\tau}\right) + \frac{1}{\tau^2}\left\{e^{jW\tau} - e^{-jW\tau}\right)\right.$$

$$= \frac{K}{2\pi\tau}(2\sin W\tau) + \frac{1}{2\pi\tau}(2\cos W\tau) + \frac{1}{2\pi W}(2\sin W\tau)$$

→ Find the ACF of the following PSDS

(i) $S_{xx}(\omega) = \frac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)}$

(ii) $S_{xx}(\omega) = \frac{8}{(9+\omega^2)^2}$

(ii) $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{8}{(9+\omega^2)^2} e^{j\omega\tau} d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{8}{36}\left[\frac{(3)(2)}{9+\omega^2}\right]^2 e^{j\omega\tau} d\omega$$

we have $e^{-a|\tau|} \xrightarrow{F.T} \frac{2a}{a^2+\omega^2}$

Inverse transform of $\left(\frac{2a}{\omega^2+a^2}\right)^2$ is convolution of $e^{-a|\tau|}$ with $e^{-a|\tau|}$

ie $\left(\frac{2a}{a^2+\omega^2}\right)^2 \xrightarrow{F^{-1}} \int_{-\infty}^{\infty} e^{-a|\tau|} e^{-a|t-\tau|} d\tau$

There are two cases on above integration

case i) If $t>0$ then

$$= \int_{-\infty}^{0} e^{-a(-\tau+t-\tau)} d\tau + \int_{0}^{t} e^{-a(\tau+t-\tau)} d\tau + \int_{t}^{\infty} e^{-a(\tau-t-\tau)} d\tau$$

since $|\tau| = \tau$ $\tau>0$ and $|t-\tau| = t-\tau$

for $t-z>0$ ie $\tau<t$

$= -\tau$ $\tau<0$ $|t-\tau| = -(t-\tau)$

for $-(t-\tau)<0$ ie $z<T$

$$= e^{-at}\left[\frac{e^{2a\tau}}{2a}\right]_{0}^{\infty} + e^{-at}(\tau]_0^t + e^{at}\left[\frac{e^{-2a\tau}}{-2a}\right]_t^{\infty}$$

$$= \frac{e^{-at}}{a}(1+ta)$$

case ii) If $t<0$ $e^{-a|t|} + e^{-a|t|} = \frac{e^{-at}}{a}(1-ta)$

$$R_{xx}(\tau) = \frac{e^{-a|\tau|}}{a}(1+|\tau|a)$$

(ii) $S_{xx}(\omega) = \dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} = \dfrac{157}{(16+\omega^2)(9+\omega^2)} + \dfrac{12\omega^2}{(16+\omega^2)(9+\omega^2)}$.

$R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)] = F^{-1}\left\{ \dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} e^{j\omega\tau} d\omega \right\}$

consider $\dfrac{157+12\omega^2}{(16+\omega^2)(9+\omega^2)} = \dfrac{A}{16+\omega^2} + \dfrac{B}{9+\omega^2}$

$157+12\omega^2 = A(9+\omega^2) + B(16+\omega^2)$

put $\omega^2 = -16$ $\quad -35 = A(9-16) \Rightarrow A = 5$

put $\omega^2 = -9$ $\quad 49 = B(16-9) = B = 7$

$R_{xx}(\tau) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\left(\dfrac{5}{16+\omega^2} + \dfrac{7}{9+\omega^2}\right) e^{j\omega\tau} d\omega$

$= \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{5}{16+\omega^2} e^{j\omega\tau} d\omega + \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{7}{9+\omega^2} e^{j\omega\tau} d\omega$

$= 5\,\dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{1}{16+\omega^2} e^{j\omega\tau} d\omega + 7\,\dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{1}{9+\omega^2} e^{j\omega\tau} d\omega$

$= \dfrac{5}{2\times4}\left[\dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{2\times4}{16+\omega^2} e^{j\omega\tau} d\omega\right] + \dfrac{7}{2\times3}\left[\dfrac{1}{2\pi}\int_{-\infty}^{\infty} \dfrac{2\times3}{9+\omega^2} e^{j\omega\tau} d\omega\right]$

$= \dfrac{5}{8} F^{-1}\left[\dfrac{2\times4}{16+\omega^2}\right] + \dfrac{7}{6} F^{-1}\left[\dfrac{2\times3}{9+\omega^2}\right]$

$= \dfrac{5}{8} e^{-4|\tau|} + \dfrac{7}{6} e^{-3|\tau|}$.

(iii) $S_{xx}(\omega) = \dfrac{8}{(9+\omega^2)^2} = \dfrac{4}{3}\cdot\dfrac{2\times3}{(9+\omega^2)^2} = \dfrac{4}{3\times6}\cdot\dfrac{(6)^2}{(9+\omega^2)^2}$

$= \dfrac{2}{9}\left[\dfrac{2\times3}{9+\omega^2}\right]^2$

$R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)] = F^{-1}\left[\dfrac{2}{9}\left(\dfrac{2\times3}{9+\omega^2}\right)^2\right\}$

$= \dfrac{2}{9}\left[e^{-3|\tau|} * e^{-3|\tau|}\right]$

$= \dfrac{2}{9}\int_{-\infty}^{\infty} e^{-3|\tau_1|} e^{-3|\tau-\tau_1|} d\tau_1$

$= \dfrac{2}{9}\int_{-\infty}^{\infty} e^{-3[|\tau_1| + |\tau-\tau_1|]} d\tau_1$

case (i) $\tau > 0$:

$$R_{xx}(\tau) = \frac{2}{9}\int_{-\infty}^{0} e^{-3\{|\tau_1| + |\tau-\tau_1|\}}d\tau_1 + \int_{0}^{\tau} e^{-3[|\tau_1| + |\tau-\tau_1|]}d\tau_1 + \int_{\tau}^{\infty} e^{-3[|\tau_1| + |\tau-\tau_1|}d\tau_1 - I$$

$$|\tau_1| = \begin{cases} \tau_1 & \text{for } \tau_1 > 0 \\ -\tau_1 & \text{for } \tau_1 < 0 \end{cases} \qquad |\tau-\tau_1| = \begin{cases} \tau-\tau_1 & \tau-\tau_1 > 0 \\ & \tau_1 < \tau \\ -(\tau-\tau_1) & \tau_1 > \tau \end{cases}$$

$$R_{xx}(\tau) = \frac{2}{9}\left[\int_{\tau_1=-\infty}^{0} e^{-3[-\tau_1 + (\tau-\tau_1)]}d\tau_1 + \int_{\tau_1=0}^{\tau} e^{-3(\tau_1+\tau-\tau_1)}d\tau_1 + \int_{\tau_1=\tau}^{\infty} e^{-3(\tau_1-(\tau-\tau_1))}d\tau_1\right.$$

consider $\frac{2}{9}\int_{\tau_1=-\infty}^{0} e^{3\tau_1 - 3\tau + 3\tau_1}d\tau_1 = \frac{2}{9}\int_{\tau_1=-\infty}^{0} e^{6\tau_1 - 3\tau}d\tau_1$

$$= \frac{2}{9}e^{-3\tau}\int_{\tau_1=-\infty}^{0} e^{6\tau_1}d\tau_1 = \frac{2}{9}e^{-3\tau}\cdot\left.\frac{e^{6\tau_1}}{6}\right|_{\tau_1=-\infty}^{0} = \frac{e^{-3\tau}}{27}$$

consider $\int_{\tau_1=0}^{\tau} e^{-3\tau_1 + 3\tau + 3\tau_1}d\tau_1 = \int_{\tau_1=0}^{\tau} e^{-3\tau}d\tau_1 = e^{-3\tau}\cdot\tau.$

consider $\int_{\tau_1=\tau}^{\infty} e^{-3\tau_1 + 3\tau - 3\tau_1}d\tau_1 = e^{3\tau}\int_{\tau_1=\tau}^{\infty} e^{-6\tau_1}d\tau_1$

$$e^{3\tau}\left.\frac{e^{-6\tau_1}}{-6}\right|_{\tau}^{\infty} = \frac{e^{-3\tau}}{6}$$

$$R_{xy}(\tau) = \underline{\underline{\frac{2}{27}e^{-3\tau}}} + \underline{\underline{\frac{2}{9}e^{-3\tau}\tau}}$$

→ Two independent stationary r.p $x(t)$ and $y(t)$ have power spectrum densities $S_{xx}(\omega) = \frac{16}{\omega^2+16}$ and $S_{yy}(\omega) = \frac{\omega^2}{\omega^2+16}$ respectively with zero mean. Let another r.p $U(t) = x(t) + y(t)$

Then find out (i) psd of $u(t)$ (ii) $S_{xy}(\omega)$ (iii) $S_{xu}(\omega)$.

Sol: Given $x(t)$ and $y(t)$ are independent random processes

$S_{xx}(\omega) = \frac{16}{\omega^2+16}$ and $S_{yy}(\omega) = \frac{\omega^2}{\omega^2+16}$ and

$u(t) = x(t) + y(t)$

(i) Power density of $u(t)$ is $S_{uu}(\omega) = S_{xx}(\omega) + S_{yy}(\omega)$

$$S_{uu}(\omega) = \frac{16}{\omega^2+16} + \frac{\omega^2}{\omega^2+16} = 1$$

(ii) since $x(t)$ and $y(t)$ independent and uncorrelated fun

$$\therefore S_{xy}(\omega) = 2\pi \bar{x}\bar{y}\,\delta(\omega) = 0$$

(iii) $S_{xu}(\omega) =$ F.T of $R_{xu}(T)$

$$R_{xu}(T) = E[x(t)\,u(t+T)]$$

$$= E[x(t)\cdot(x(t+T) + y(t+T))]$$

$$= E[x(t)\cdot x(t+T)] + E[x(t)\cdot y(t+T)]$$

$$= R_{xx}(T) + R_{xy}(T)$$

$S_{xu}(\omega) =$ F.T of $[R_{xx}(T) + R_{xy}(T)]$

$$= S_{xx}(\omega) + S_{xy}(\omega)$$

$$= \frac{16}{\omega^2+16} + 0$$

$$S_{xu}(\omega) = \frac{16}{\omega^2+16}$$