UNIT-3

Joint Distribution function:- The joint probability distribution function is defined as the probability of joint event $\{X \le x, Y \le y\}$, which is a function of the numbers x & y. Joint probability distribution function is denoted by the symbol $F_{XY}(x,y)$. Hence

$$F_{XY}(x,y) = P\{X \le x, Y \le y\} \quad -①$$

Joint probability distribution fun of discrete Random variables:-
Let X and Y are two discrete random variables defined on the sample space "S". Let "X" has "N" possible values of x_m and Y has M possible values of y_m then $F_{XY}(x,y)$ is denoted as

$$F_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_m, y_m)\, u(x-x_m)\, u(y-y_m) \quad -②$$

Here $P(x_m, y_m)$ denotes the prob of joint event $\{X = x_m, Y = y_m\}$ & $u(x-x_m)$, $u(y-y_m)$ denotes the unit step function.

Properties of Joint distribution:-

- $F_{XY}(-\infty,-\infty)=0$
  $F_{XY}(-\infty, y) = 0$
  $F_{XY}(x, -\infty) = 0.$
- $F_{XY}(\infty, \infty) = 1$
- $0 \le F_{XY}(x,y) \le 1$
- $F_{XY}(x,\infty) = F_X(x)$ &
  $F_{XY}(\infty,y) = F_Y(y)$
- $F_{XY}(x,y)$ is a non-decreasing function of x and y.
- $P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}$
  $= F_{XY}(x_2,y_2) + F_{XY}(x_1,y_1) - F_{XY}(x_2,y_1) - F_{XY}(x_1,y_2).$

Marginal distribution functions:-

If $F_{XY}(x,y)$ is the joint probability distribution of X and Y. Then the distribution function of one random variable can be obtained by setting the value of other random variable to "∞" in joint distribution

$$F_Y(y) = F_{XY}(\infty, y) \quad (③)$$
$$F_X(x) = F_{XY}(x, \infty).$$

Proof :- Let us consider two events $A = \{X \le x\}$ and $B = \{Y \le y\}$ joint distribution function

$$F_{XY}(x, y) = P\{X \le x, Y \le y\} = P(A \cap B)$$

now $F_{XY}(x, \infty) = P\{X \le x, Y \le \infty\} = P(A \cap B)$

here $B = \{Y \le \infty\} \Rightarrow B = S \Rightarrow A \cap B = A \cap S = A$.

$$F_{XY}(x, \infty) = P\{X \le x, Y \le \infty\} = P(A)$$
$$= P\{X \le x\} = F_X(x).$$

$\therefore F_X(x) = F_{XY}(x, \infty)$.

similarly $F_{XY}(\infty, y) = P\{X \le \infty, Y \le y\} = P(A \cap B)$.

here $A = \{X \le \infty\} = S$

$\Rightarrow A \cap B = S \cap B = B$.

$$F_{XY}(\infty, y) = P\{X \le \infty, Y \le y\} = P(A \cap B) = P(B).$$
$$= P\{Y \le y\} = F_Y(y).$$

$\therefore F_Y(y) = F_{XY}(\infty, y)$.

→ The joint space for two random variable X & Y corresponding probabilities are shown in table.

x, y:	1,1	2,2	3,3	4,4
P:	0.2	0.3	0.35	0.15

Find and plot (i) $F_{XY}(x, y)$ (ii) Marginal distribution function of X and Y ie $F_X(x)$ and $F_Y(y)$ (iii) Find prob of $P\{X \le 2, Y \le 2\}$ and (iv) find $P\{1 < X \le 3, Y \ge 3\}$.

Sol :- we know that joint distribution function

$$F_{XY}(x, y) = P\{X \le x, Y \le y\}$$

$$= \sum_{x_i \le x} \sum_{y_i \le y} P(X = x_i, Y = y_i)$$

$F_{XY}(x, y)$ for $\{X \le 4, Y \le 4\}$

$$F_{XY}(x, y) = P\{X = 4, Y = 4\} + P(X = 3, Y = 3) + P(X = 2, Y = 2) + P(X = 1, Y = 1)$$
$$= 0.2 + 0.3 + 0.35 + 0.15 = 1$$

$F_{XY}(x, y)$ for $\{X \le 3, Y \le 3\}$

$$F_{XY}(x, y) = P\{X = 3, Y = 3\} + P(X = 2, Y = 2) + P(X = 1, Y = 1) = 0.2 + 0.3 + 0.35 = 0.85$$

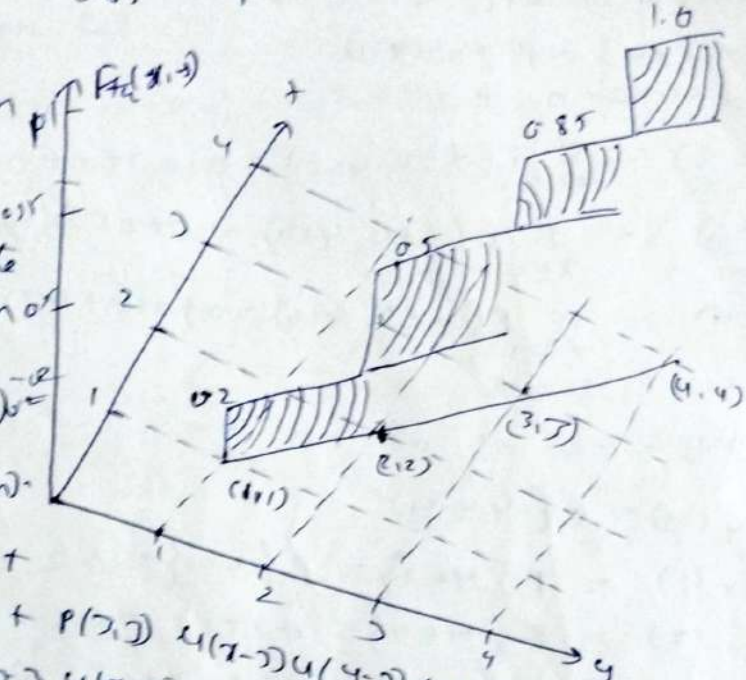
$F_{XY}(x,y)$ for $\{x \le 2, y \le 2\}$

$F_{XY}(x,y) = P(x=2, y=2) + P(x=1, y=1) = 0.2+0.3 = 0.5$

$F_{XY}(x,y)$ for $\{x \le 1, y \le 1\} = 0.2$

x, y :	1,1	2,2	3,3	4,4
P -	0.2	0.3	0.35	0.15
F_{XY} -	0.2	0.5	0.85	1

Fig :- Joint distribution plot of $F_{XY}(x,y)$.



(8)

Here X & Y are discrete random variables then of $F_{XY}(x,y)$

$$F_{XY}(x,y) = \sum_{n=1}^{4} \sum_{m=1}^{4} P(x_m, y_m)\, u(x-x_m)\, u(y-y_m).$$

$= P(1,1)\,u(x-1)\,u(y-1) + P(2,2)\,u(x-2)\,u(y-2) + P(3,3)\,u(x-3)\,u(y-3) + P(4,4)\,u(x-4)\,u(y-4).$

$= 0.2\,u(x-1)\,u(y-1) + 0.3\,u(x-2)\,u(y-2) + 0.35\,u(x-3)\,u(y-3) + 0.15\,u(x-4)\,u(y-4)$

(ii) Marginal distribution of X & Y :-

Marginal distribution of X :-

$P(X=1) = \sum_{y \le 4} P(X=1, Y=y)$

$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$= 0.2 + 0+0+0 = 0.2$

$P(X=2) = \sum_{y \le 4} P(X=2, Y=y)$

$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=2, Y=3) + P(X=2, Y=4)$

$= 0+0.3+0+0 = 0.3$

$P(X=3) = \sum_{y \le 4} P(X=3, Y=y) = P(X=3, Y=1) + P(X=3, Y=2) + P(X=3, Y=3) + P(X=3, Y=4) = 0+0+0.35+0 = 0.35$

$P(X=4) = 0.15$

x : 1 2 3 4

$P(x)$ = 0.2 0.3 0.35 0.15

$F_x(x)$ = 0.2 0.5 0.85 1

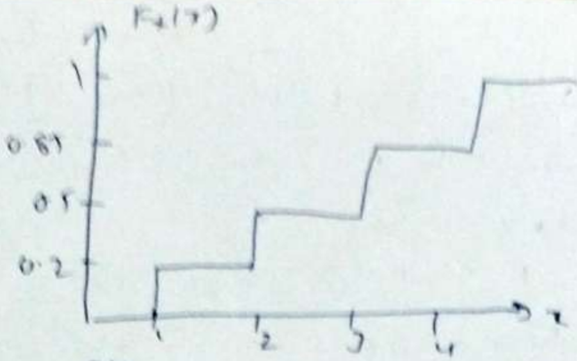


Fig. Marginal distribution of x.

Marginal distribution of y

$P(y=1) = \sum_{x \le 4} P(x \le 4, y=1)$

$= P(x=1, y=1) + P(x=2, y=1) + P(x=3, y=1) + P(x=4, y=1)$

$= 0.2+0+0+0 = 0.2$

$P(y=2) = \sum_{x \le 4} P(x \le 4, y=2) = 0+0.3+0+0 = 0.3$

$P(y=3) = \sum_{x \le 4} P(x \le 4, y=3) = 0+0+0.35+0 = 0.35$

$P(y=4) = \sum_{x \le 4} P(x \le 4, y=4) = 0+0+0+0.15 = 0.15$

y = 1 2 3 4

$p(y)$ = 0.2 0.3 0.35 0.15

$F_y(y) = P\{Y \le y\}$

$F_y(1) = P\{y \le 1\} = P\{y=1\} = 0.2$

$F_y(2) = P\{y \le 2\} = 0.2+0.3 = 0.5$

$F_y(3) = P\{y \le 3\} = 0.2+0.3+0.35 = 0.85$

$F_y(4) = 1$

y = 1 2 3 4

$P(y)$ = 0.2 0.3 0.35 0.15

$F_y(y)$ = 0.2 0.5 0.85 1.0

Marginal distribution of y is similar to marginal of x

(ii) $P(x \le 2, y \le 2) = \cancel{P(x=1)}$

$= P(x=1, y=1) + P\{(x=2, y=2)\}$

$= 0.2+0.3 = 0.5$

(iii) $P(1 < x \le 3, y \ge 3)$

$P(x=2, y=3) + P(x=3, y=3) + P(x=2, y=4) +$

$P(x=3, y=4) = \underline{0.35}$

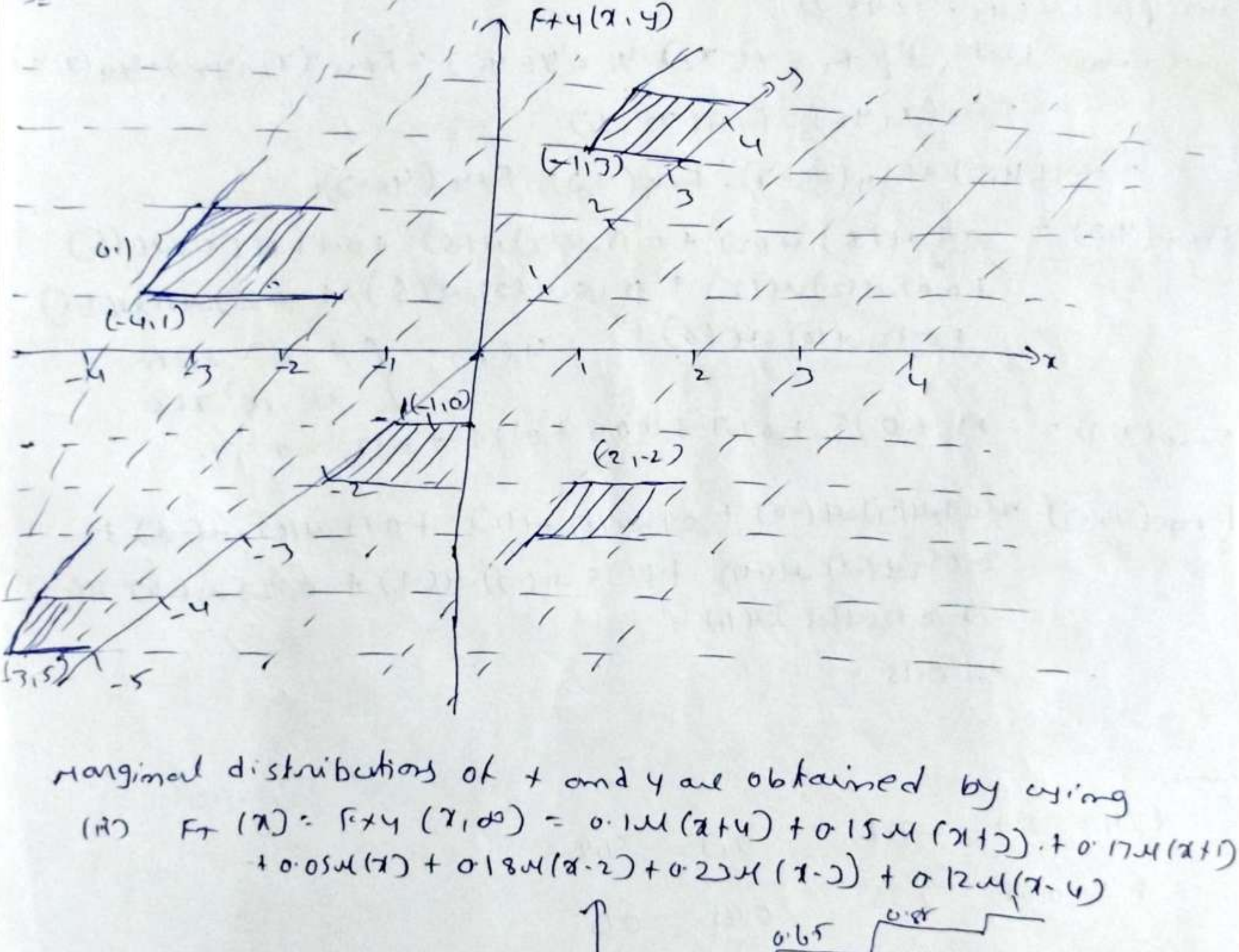
→ The joint space for two random variable X & Y and the corresponding probabilities as shown on table. Find and plot (i) $F_{XY}(x,y)$ (ii) Marginal distributions of X and Y (iii) Find $P(0.5 \le X \le 1.5)$ (iv) Find $P(X \le 1, Y \le 2)$ and $P(1 \le X \le 2, Y \le 3)$.

→ Discrete random variables X & Y have a joint distribution function

$$F_{XY}(x,y) = 0.1u(x+4)u(y-1) + 0.15u(x+3)u(y+5) + 0.17u(x+1)u(y-3) + 0.05u(x)u(y-1) + 0.18u(x-2)u(y+2) + 0.23u(x-3)u(y-4) + 0.12u(x-4)u(y+3).$$

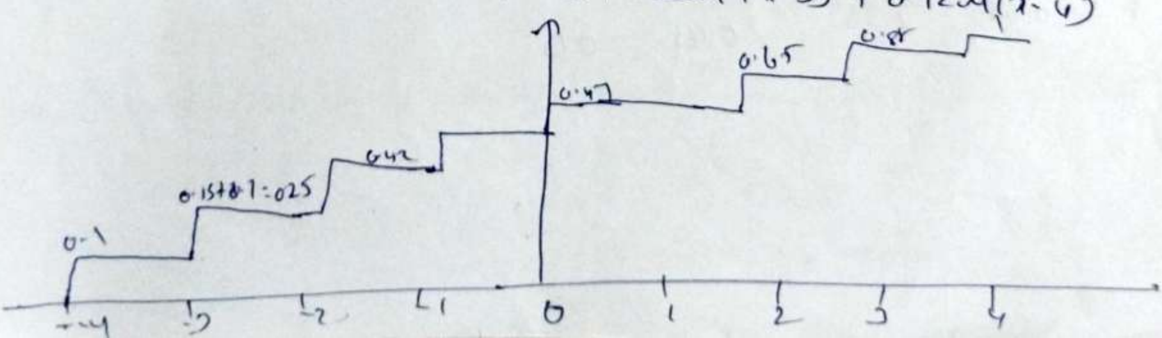
Find (i) sketch $F_{XY}(x,y)$ (ii) marginal distributions of X and Y (iii) $P(-1 < X \le 4, -3 < Y \le 3)$ (iv) Find $P(X < 1, Y \le 2)$.

Sol



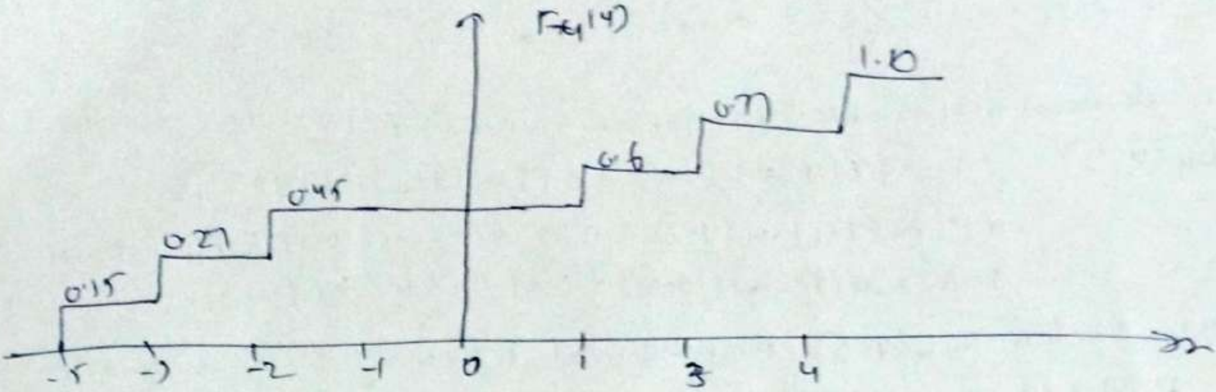
Marginal distributions of X and Y are obtained by using

(ii) $F_X(x) = F_{XY}(x,\infty) = 0.1u(x+4) + 0.15u(x+3) + 0.17u(x+1) + 0.05u(x) + 0.18u(x-2) + 0.23u(x-3) + 0.12u(x-4)$



$F_X(y) = F_{XY}(\infty, y)$

$= 0.1u(y+5) + 0.12u(y+3) + 0.18u(y+2) + 0.15u(y-1) +$
$0.17(y-3) + 0.23u(y-4)$.



(iii) $P(-1 < X \le 4, -3 < Y \le 3)$

we know that $P\{x_1 \le X \le x_2, y_1 < Y \le y_2) = F_{XY}(x_2, y_2) + F_{XY}(x_1, y_1)$
$- F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1)$

$= F_{XY}(4, 3) + F_{XY}(-1, -3) - F_{XY}(-1, 3) - F_{XY}(4, -3)$.

$F_{XY}(4,3) = 0.1u(8)u(2) + 0.15u(7)u(8) + 0.17u(5)u(6)$
$+ 0.05u(2)u(2) + 0.18u(2)u(5) + 0.23u(1)u(-1)$
$+ 0.12u(6)u(6)$

$u(x) = \begin{cases} 1 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$

$F_{XY}(4,3) = 0.1 + 0.15 + 0.17 + 0.05 + 0.18 + 0.12 = 0.77$.

$F_{XY}(-1,-3) = 0.1u(3)u(-4) + 0.15u(2)u(2) + 0.17u(0)u(-6) +$
$0.05u(-1)u(-4) + 0.18u(-3)u(-1) + 0.23u(-4)u(-3)$
$+ 0.12u(-5)u(6)$

$= 0.15$

⇒

$x, y = 1{:}1 \quad 2{,}2 \quad 3{,}3 \quad 4{,}4$

$P = 0.05 \quad 0.35 \quad 0.45 \quad 0.15$

JOINT DENSITY AND ITS PROPERTIES:-

If X & Y are two r.v.s and $F_{XY}(x,y)$ is the joint distribution function of X and Y then the joint probability density function is defined by the second derivative of the joint distribution function. It is denoted by $f_{X,Y}(x,y)$.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y} \quad -(1)$$

If X & Y are discrete r.v then the joint prob density function is given by $f_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\,\delta(x-x_n)\,\delta(y-y_m)$ -(2)

Properties:-

→ Joint density function is a non-negative $f_{XY}(x,y) \geq 0$.

→ The total area under the joint density function is always unity $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

→ Joint distribution function

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

→ Marginal distribution functions of X & Y are

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx \quad \text{and}$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$$

→ The prob. of joint event $\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\}$ is given by

$$P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy.$$

→ Marginal probability density functions $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

and $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$.

Marginal density function:

$f_X(x)$ and $f_Y(y)$ are called marginal prob density functions (or) simply marginal density functions. These are density functions of the single variable X and Y are defined as the derivatives of the marginal distributions functions

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

→ when N-random variables $X_1, X_2, \ldots X_N$ are modeled, the joint density function becomes the N-fold partial derivative of the N-dimensional distribution function

$$f_{X_1 X_2 \ldots X_N}(x_1, x_2 \ldots x_N) = \frac{\partial^N F_{X_1, X_2 \ldots X_N}(x_1, x_2, \ldots x_N)}{\partial x_1\, \partial x_2 \ldots \partial x_N}$$

By direct integration this result is:

$$F_{X_1 X_2 \ldots X_N}(x_1, x_2 \ldots x_N) = \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f_{X_1, X_2 \ldots X_N}(x_1, x_2, \ldots x_N)\, dx_1\, dx_2 \ldots dx_N$$

→ Find the marginal densities of X & Y using joint density function

$f_{XY}(x,y) = u(x) \cdot u(y)\, x e^{-x(y+1)}$

sol: Given $f_{XY}(x,y) = u(x) \cdot u(y)\, x e^{-x(y+1)}$

Marginal density of X is $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

$$f_X(x) = \int_{-\infty}^{\infty} u(x) \cdot u(y)\, x e^{-x(y+1)}\, dy = u(x) \cdot x \cdot \int_{-\infty}^{\infty} e^{-xy} e^{-x} u(y)\, dy.$$

$$= x \cdot u(x) e^{-x} \int_{-\infty}^{\infty} e^{-xy} u(y) \cdot dy = x u(x) e^{-x} \int_{0}^{\infty} e^{-xy}\, dy.$$

$$= x \cdot u(x) e^{-x} \cdot \frac{e^{-xy}}{-x} \Big|_0^{\infty} = -u(x) e^{-x} [-1] =$$

$$f_X(x) = u(x) e^{-x}$$

→ $$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_{-\infty}^{\infty} u(x)\, u(y)\, x e^{-x(y+1)}\, dx.$$

$$= u(y) \int_{-\infty}^{\infty} x\, u(x) e^{-xy} \cdot e^{-x}\, dx = u(y) \int_{0}^{\infty} x e^{-x(y+1)}\, dx.$$

$$= u(y)\left[x\cdot\frac{e^{-x(y+1)}}{-(y+1)} + \int_0^{\infty}\frac{e^{-x(y+1)}}{y+1}\cdot 1\right]$$

$$= u(y)\left[-\frac{x\,e^{-x(y+1)}}{y+1} + \frac{e^{-x(y+1)}}{(y+1)^2}\right] = \frac{u(y)}{(y+1)^2}$$

$$\therefore f_y(y) = \frac{u(y)}{(y+1)^2} \;//.$$

→ Find the marginal density of X & Y using joint density $f_{xy}(x,y)$

$$f_{xy}(x,y) = 2u(x)\,u(y)\,\exp\left[-\left(4y+\frac{x}{2}\right)\right]$$

sol^n: $f_{xy}(x,y) = 2u(x)\cdot u(y)\,e^{-4y-\frac{x}{2}}$

Marginal density of X is $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$

$$= \int_{-\infty}^{\infty} 2u(x)\,u(y)\,e^{-4y}\,e^{-x/2}\,dy = 2u(x)e^{-x/2}\int_{-\infty}^{\infty} u(y)e^{-4y}\,dy$$

$$= 2u(x)e^{-x/2}\int_0^{\infty} e^{-4y}\,dy = 2u(x)e^{-x/2}\cdot\frac{e^{-4y}}{-4}\Big|_0^{\infty} = \frac{u(x)e^{-x/2}}{2}.$$

Marginal density of Y is $f_y(y) = \int_{-\infty}^{\infty} 2u(x)\,u(y)\,e^{-4y}\,e^{-x/2}\,dx.$

$$= 2u(y)\,e^{-4y}\int_{-\infty}^{\infty} u(x)\,e^{-x/2}\,dx = 2u(y)e^{-4y}\int_0^{\infty} e^{-x/2}\,dx$$

$$= 2u(y)e^{-4y}\,\frac{e^{-x/2}}{-1/2}\Big|_0^{\infty} = 4u(y)e^{-4y}.$$

<u>conditional distribution and density functions</u>:-

The conditional distribution of a random variable X given some event B is given by $F_x(x|B) = P\{X \le x|B\}$

$$= \frac{P\{X \le x \cap B\}}{P(B)} = P(X \le x, B)/P(B) \quad \text{if } P(B) \ne 0 \;-(1).$$

The conditional density function $f_x(x|B) = \frac{d}{dx}F_x(x|B) \;-(2)$

<u>conditional distribution and density at Point conditioning</u>

Let the event B is defined at specific value of (Point condition) is given by $B = \{y_0 - \Delta y < Y \le y_0 + \Delta y\}$ here Δy is a very less value i.e $\Delta y \to 0$

$$F_x(x|B) = F_x(x|(y_0 - \Delta y < y \le y_0 + \Delta y))$$

$$F_X(x|A) = \frac{P(X \le x \cap (y_0 - \Delta y < Y \le y_0 + \Delta y))}{P(y_0 - \Delta y < Y \le y_0 + \Delta y)} = \frac{P(X \le x,\ y_0 - \Delta y < Y \le y_0 + \Delta y)}{P(y_0 - \Delta y < Y \le y_0 + \Delta y)}$$

$$= \frac{F_{XY}(x,\ y_0 - \Delta y < y \le y_0 + \Delta y)}{P(y_0 - \Delta y < y \le y_0 + \Delta y)}$$

$$= \frac{\int_{y_0 - \Delta y}^{y_0 + \Delta y} \int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy}{\int_{y_0 - \Delta y}^{y_0 + \Delta y} f_Y(y)\,dy} \quad -③$$

case i) - both are continuous random variables :-

$$\Rightarrow F_X(x / y_0 - \Delta y < Y \le y_0 + \Delta y) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\,dx \cdot (2\Delta y)}{f_Y(y)\,(2\Delta y)} \quad -⑩$$

as $\Delta y \to 0$

$$F_X(x / Y = y_0) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\,dx}{f_Y(y)} \quad -④$$

Apply differentiation on both sides

$$\Rightarrow f_X(x/Y=y_0) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad -⑤ \qquad \Rightarrow f_Y(y/x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

case 2: If both X and Y are discrete random variables with value x_i and y_j respectively. where $i = 1, 2, 3 \ldots N$, $j = 1, 2, 3 \ldots N$.
$P(x_i)$ and $P(y_j)$ are the corresponding prob and $P(x_i, y_j)$ denotes the joint occurance of x_i and y_j

$$f_X(x) = \sum_{i=1}^{N} P(x_i)\,\delta(x - x_i) \quad -⑥$$

$$f_Y(y) = \sum_{j=1}^{M} P(y_j)\,\delta(y - y_j) - ⑦$$

and
$$f_{XY}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{M} P(x_i, y_j)\,\delta(x - x_i)\,\delta(y - y_j) \quad -⑧$$

at y_k is the value of Y term

$$F_X(x / Y = y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)}\,u(x - x_i) \quad -⑨$$

$P(y_k) \ne 0$

apply differentiating on both sides

$$f_X(x/Y=y_K) = \sum_{i=1}^{N} \frac{P(x_i, y_K)}{P(y_K)} \delta(x-x_i) \quad -(10)$$

conditional distribution and density - interval conditions:

Let the event "B" in terms of R.V "Y" is given by $B = \{ y_a \le Y \le y_b \}$ - (11)

here y_a and y_b are real numbers

$$P(B) = P\{ y_a < Y \le y_b \} \neq 0 \quad -(12)$$

$$F_X(x/y_a < Y \le y_b) = \frac{F_{XY}(x, y_b) - F_{XY}(x, y_a)}{F_Y(y_b) - F_Y(y_a)}$$

$$= \frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy}{\int_{y_a}^{y_b} f_Y(y)\,dy}$$

(or)

$$= \frac{\int_{y_a}^{y_b}\int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

differentiating on both sides

$$f_X(x/y_a < Y \le y_b) = \frac{\int_{y_a}^{y_b} f_{XY}(x,y)\,dy}{\int_{y_a}^{y_b}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy}$$

NOTE: conditional distribution & density functions:

-> For continuous random variables

$$F_X(x/Y=y_0) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\,dx}{f_Y(y)} \quad \text{and}$$

$$f_X(x/Y=y_0) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

For discrete R.Vs $F_X(x/y_K) = \sum_{i=1}^{N} \frac{P(x_i, y_K)}{P(y_K)} u(x-x_i)$ &

$$f_X(x/y_K) = \sum_{i=1}^{N} \frac{P(x_i, y_K)}{P(y_K)} \delta(x-x_i).$$

→ The joint prob density function of two R.Vs X & Y is given by

$$f(x,y) = \begin{cases} a(2x+y^2) & 0 \le x \le 2,\ 2 \le y \le 4 \\ 0 & \text{else} \end{cases}$$

and find (i) value of a (ii) $P(X \le 1, Y > 3)$

Sol: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

$\therefore \int_{y=2}^{4}\int_{x=0}^{2} a(2x+y^2)\,dx\,dy = 1$

$a \int_{y=2}^{4}\int_{x=0}^{4} 2x+y^2\,dx\,dy = 1 \Rightarrow a\int_{y=2}^{4} \left[x^2+xy^2\right]_0^2\Big]\,dy = 1$

$\Rightarrow a\int_{y=2}^{4} \left[4+2y^2\right] dy = 1 \Rightarrow a\left[4y+\frac{2}{3}y^3\right]_2^4 = 1$

$\Rightarrow a\left[4(2)+\frac{2}{3}\left[64-8\right]\right] = 1 \qquad a = \frac{3}{136}$.

$f(x,y) = a(2x+y^2) = \frac{3}{136}(2x+y^2) \quad 0 \le x \le 2,\ 2 \le y \le 4$
$\qquad = 0 \quad$ else

(ii) $P\{X \le 1, Y > 3\}$

$= \int_{y=3}^{4}\int_{x=0}^{1} f_{XY}(x,y)\,dx\,dy = \frac{3}{136}\int_{y=3}^{4}\int_{x=0}^{1} 2x+y^2\,dx\,dy$

$= \frac{3}{136}\int_3^4 \left.x^2+xy^2\right|_0^1 dy = \frac{3}{136}\int_3^4 1+y^2\,dy = \frac{3}{136}\left\{y+\frac{y^3}{3}\right\}_3^4$

$= \frac{3}{136}\left[1+\frac{64-27}{3}\right] = \frac{3}{136}\left[1+\frac{37}{3}\right] = \frac{5}{17}$.

→ The joint prob density function of 2 R.Vs X, Y is given by

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{else} \end{cases}$$

(i) The value of c (ii) marginal distribution

Sol: $\int_{y=0}^{2}\int_0^1 c(2x+y)\,dx\,dy = 1 \Rightarrow c\int_0^2 \left[x^2+xy\right]_0^1 dy = 1$

$= c\int_0^2 1+y\,dy = 1 \Rightarrow c\left\{y+\frac{y^2}{2}\right\}_0^2 = 1 \quad c = \frac{1}{4}$

(ii) marginal distribution of X is given by

$F_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy.$

$$F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty}\left(\frac{2x+y}{4}\right) dx\,dy = \int_{x=0}^{x}\int_{0}^{2}\left(\frac{x}{2}+\frac{y}{4}\right) dy\,dx.$$

$$= \int_{x=0}^{x} \frac{xy}{2} + \frac{y^2}{8}\Big|_0^2 dx = \int_{x=0}^{x}\left(x+\frac{1}{2}\right) dx = \frac{x^2}{2}+\frac{1}{2}x\Big|_{x=0}^{x}$$

$$= \frac{x^2+x}{2}.$$

Marginal distribution of y is given by

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = \int_0^y\int_0^1 \frac{1}{4}(2x+y)\,dx\,dy$$

$$= \frac{1}{4}\int_0^y (x^2+xy)\Big|_0^1 dy = \frac{1}{4}\int_0^y (1+y)\,dy = \frac{1}{4}\left[y+\frac{y^2}{2}\right] = \frac{2y+y^2}{8}.$$

→ The random variable X & Y have the joint prob density function

$$f_{XY}(x,y) = \begin{cases} Ae^{-(2x+y)} & x,y \geq 0 \\ 0 & \text{else} \end{cases}$$

(i) evaluate A $= 2$

(ii) marginal distributions $F_X(x) = 1-e^{-2x}$ $F_Y(y) = 1-e^{-y}$

(iii) " densities of X & Y $f_X(x) = 2e^{-x}$ $f_Y(y) = e^{-y}$

(iv) Find the joint CDF $\{(1-e^{-2x})(1-e^{-y})\}$

→ ~~Fin~~ Joint density function of two dimensional R.V X, Y is given by

$$f(x,y) = \begin{cases} \frac{8}{9}xy & 1 \leq x \leq y \leq 2 \\ 0 & \text{else} \end{cases}$$

Find marginal densities of X, Y.

<u>Sol</u>: marginal density of X is given by $f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$.

$$= \int_{y=x}^{2} \frac{8}{9}xy\,dy = \frac{8}{9}\left[\frac{xy^2}{2}\right]_x^2 = \frac{8}{9}\left[\frac{4x}{2}-\frac{x^3}{2}\right]$$

$$= \frac{4}{9}\left[4x-x^3\right]$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_1^y \frac{8}{9}xy = \frac{8}{9}\left[\frac{x^2y}{2}\right]_1^y$$

$$= \frac{4}{9}\left[y^3-y\right].$$

⇒ Find the conditional density functions $f_X(x/y_1)$, $f_X(x/y_2)$, $f_Y(y/x_1)$, $f_Y(y/x_2)$ for the joint function defined by $P(x_1,y_1) = \frac{2}{15}$, $P(x_2,y_2) = \frac{2}{15}$ $P(x_2,y_1) = \frac{1}{15}$

$P(x_2, y_3) = \frac{5}{15}, \quad P(x_1, y_2) = 4/15$

$\underline{\text{Sly}}$.. $P(x_1) = P(x_1, y_1) + P(x_1, y_3) = \frac{2}{15} + \frac{4}{15} = \frac{2}{5}.$

$P(x_2) = P(x_2, y_2) + P(x_2, y_1) + P(x_2, y_3) = \frac{3}{15} + \frac{1}{15} + \frac{5}{15} = \frac{9}{15}.$

$$P(x, y) = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & \\ x_1 & \frac{2}{15} & 4/15 & - & \frac{6}{15} \\ x_2 & \frac{1}{15} & \frac{3}{15} & \frac{5}{15} & \frac{9}{15} \\ \hline & \frac{3}{15} & \frac{7}{15} & \frac{5}{15} & \end{array}$$

(i) $f_+(x/y_1) = \sum_{i=1}^{2} \frac{P(x_i, y_1)}{P(y_1)} \delta(x - x_i) = \sum_{i=1}^{2} \frac{P(x_i, y_1)}{\frac{3}{15}} \delta(x - x_i)$

$= \frac{15}{3}\left[P(x_1 y_1)\delta(x - x_1) + P(x_2 y_1)\delta(x - x_2)\right].$

$= \frac{15}{3}\left[\frac{2}{15}\delta(x - x_1) + \frac{1}{15}\delta(x - x_2)\right]$

(ii) $f_+(x/y_2) = \sum_{i=1}^{2} \frac{P(x_i, y_2)}{P(y_2)} \delta(x - x_i) = \sum_{i=1}^{2} \frac{P(x_i, y_2)}{\frac{1}{15}} \delta(x - x_i).$

$= \frac{7}{15}\left[P(x_1, y_2)\delta(x - x_1) + P(x_2, y_2)\delta(x - x_2)\right]$

$= \frac{7}{15}\left[\frac{4}{15}\delta(x - x_1) + \frac{7}{15}\delta(x - x_2)\right].$

(iii) $f_+(y/x_1) = \sum_{i=1}^{3} \frac{P(x_1 y_i)}{P(x_1)} \delta(y - y_i) = \sum_{i=1}^{3} \frac{P(x_1, y_i)}{\frac{6}{15}} \delta(y - y_i)$

$= \frac{15}{6}\left[P(x_1 y_1)\delta(y - y_1) + P(x_1 y_2)\delta(y - y_2)\right] +$
$P(x_1, y_3)\delta(y - y_3)]$

$= \frac{15}{6}\left[\frac{2}{15}\delta(y - y_1) + \frac{1}{15}\delta(y - y_2)\right].$

(iv) $f_+(y/x_2) = \sum_{i=1}^{3} \frac{P(x_2, y_i)}{P(x_2)} \delta(y - y_1)$

$= \sum_{i=1}^{3} \frac{P(x_2, y_i)}{\frac{9}{15}} \delta(y - y_i)$

$= \frac{15}{9}\sum_{i=1}^{3} \left[P(x_2, y_1) + P[x_2, y_2) + P(x_2, y_3)\right]$

$= \frac{15}{9}\left[\frac{1}{15}\delta(y - y_1) + \frac{9}{15}\delta(y - y_2) + \frac{5}{15}\delta(y - y_3)\right).$

Statistical independence :-

consider two events A & B are statistically independent then

$$P(A \cap B) = P(A) \cdot P(B) \quad \text{--- (1)}$$

This condition can be used to apply two random variables X and Y with events $A = \{X \le x\}$ and $B = \{Y \le y\}$ for two real numbers x and y. The two random variables are said to be statistically independent if and only if

$$P(X \le x, Y \le y) = P(X \le x) \cdot P(Y \le y) \quad \text{--- (2)}$$

The distribution function $F_{XY}(x,y) = P\{X \le x, Y \le y\}$

$$= P\{X \le x\} \, P\{Y \le y\}.$$

$$\therefore \boxed{F_{XY}(x,y) = F_X(x) \cdot F_Y(y)} \quad \text{--- (3)}$$ If X and Y are independent.

differentiate on both sides w.r.t x and y

$$\frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = \frac{d}{dx} F_X(x) \cdot \frac{d}{dy} F_Y(y)$$

$$\Rightarrow \boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)} \quad \text{--- (4)}$$ if X & Y are independent

conditional distribution and density for independent Random variables

Let $B = \{Y \le y\}$

$$F_X(x|B) = F_X(x/Y \le y) = P\{X \le x / Y \le y\}$$

$$= \frac{P\{X \le x, Y \le y\}}{P\{Y \le y\}} = \frac{F_{XY}(x,y)}{F_Y(y)} = \frac{F_X(x) F_Y(y)}{F_Y(y)}$$

$$\therefore \boxed{F_X(x/Y \le y) = F_X(x)} \quad \text{--- (5)}$$

In other word the conditional distribution function ceases to be conditional and equals to the marginal distribution for independent random variables

similarly $B = \{X \le x\}$

$$F_Y(y|B) = F_Y(y/X \le x) = P\{Y \le y / X \le x\}$$

$$F_Y(y / X \le x) = \frac{P\{Y \le y, X \le x\}}{P\{X \le x\}} = \frac{F_{XY}(x,y)}{F_X(x)} = \frac{\cancel{F_X(x)}\, F_Y(y)}{\cancel{F_X(x)}}$$

$$\boxed{F_Y(y / X \le x) = F_Y(y)} \quad (6)$$

differentiating (5) & (6) we get the conditional density functions as given by

$$f_X(x / Y \le y) = f_X(x)$$

and

$$f_Y(y / X \le x) = f_Y(y)$$

NOTE. conditional density functions are also obtained as

Let $B = \{Y \le y\}$

$$f_X(x|B) = f(x / Y \le y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}}$$

$$\boxed{f_X(x|B) = f_X(x)}$$

similarly $B = \{X \le x\}$

$$f_Y(y|B) = f_Y(y / X \le x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\cancel{f_X(x)}\, f_Y(y)}{\cancel{f_X(x)}}$$

$$\boxed{f_Y(y|B) = f_X(x)}$$

→ The joint density of two random variables X and Y is $f_{XY}(x,y) = \frac{1}{12} u(x)\, u(y)\, e^{-(x/4)-(y/3)}$ determine if X and Y are statistically independent or not.

Sol:

$$f_{XY}(x,y) = \frac{1}{12} u(x)\, u(y)\, e^{-(x/4)-(y/3)}.$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{-\infty}^{\infty} \frac{1}{12} u(x) u(y) e^{-(x/4)-(y/3)}\, dy$$

$$= \frac{1}{12} u(x) \int_0^{\infty} e^{-x/4} \cdot e^{-y/3}\, dy = \frac{1}{12} u(x) e^{-x/4} \int_0^{\infty} e^{-y/3}\, dy$$

$$= \frac{1}{12} u(x) e^{-x/4} \left. \frac{e^{-y/3}}{-1/3} \right|_0^{\infty} = \frac{1}{12} u(x) e^{-x/4} [-3(-1)]$$

$$= \frac{1}{4} u(x) e^{-x/4}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\infty}^{\infty} \tfrac{1}{12}\, u(x)\, u(y)\, e^{-x/4} e^{-y/3}\,dx.$$

$$= \tfrac{1}{12}\, u(y)\, e^{-y/3} \int_{0}^{\infty} e^{-x/4}\,dx$$

$$= \tfrac{1}{12}\, u(y)\, e^{-y/3} \left.\frac{e^{-x/4}}{-\frac{1}{4}}\right|_0^{\infty} = \tfrac{1}{12}\, u(y) e^{-y/3} (-4)(-1)$$

$$= \tfrac{1}{3}\, u(y)\, e^{-y/3}$$

$$f_X(x) \cdot f_Y(y) = \tfrac{1}{12}\, u(x)\, u(y)\, e^{-x/4} e^{-y/3} = f_{XY}(x,y).$$

Hence X and Y are independent.

→ Let X and Y be joint continuous R.Vs with joint pdf

$$f_{XY}(x,y) = x^2 + \frac{xy}{3} \quad \text{for } 0 \le x \le 1,\ 0 \le y \le 2$$
$$= 0 \quad \text{elsewhere}$$

(a) check X and Y for independence
(b) are $f(x|y)$ and $f(y|x)$ valid pdfs.

sol

$$f_X(x) = \int_0^2 f_{XY}(x,y)\,dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right)dy = \left[x^2 y + \frac{xy^2}{6}\right]_0^2 = 2x^2 + \frac{2x}{3}$$

$$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right)dx = \frac{1}{3} + \frac{y}{6}$$

since $f(x,y) \ne f(x) \cdot f(y)$ X, Y are not independent.

(b) $$f(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{x^2 + \frac{xy}{3}}{\frac{1}{3} + \frac{y}{6}}$$

consider $$\int_0^1 f(x|y)\,dx = \frac{1}{\frac{1}{3}+\frac{y}{6}} \int_0^1 \left(x^2 + \frac{xy}{3}\right)dx = \frac{1}{\frac{1}{3}+\frac{y}{6}} \left[\frac{x^3}{3} + \frac{x^2 y}{6}\right]_0^1 = 1$$

Hence $f(x|y)$ is valid pdf

$$f(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{x^2 + \frac{xy}{3}}{f_X(x)} = \frac{x^2 + xy}{2x^2 + \frac{2x}{3}}$$

consider $$\int_0^2 f(y|x)\,dy = \frac{1}{2x^2 + \frac{2x}{3}} \int_0^2 \left(x^2 + \frac{xy}{3}\right)dy = 1$$

Hence $f(y|x)$ is valid pdf

→ Joint density of X, Y is given by $f_{XY}(x,y) = 8xy\, e^{-(x^2+y^2)}$ $x \ge 0,\ y \ge 0$
$= 0$ else

Cal. the whether X & Y are independent
Find the conditional density of X given Y = y {i.e. $f_X(x|Y=y)$}

Distribution and density of a sum of random variables..

Sum of two random variables :-

Let x and y are two independent random variables and 'w' be a random variable equals to the sum of two independent random variables x and y ie $w = x+y$.

→ Here let "x" represent a random signal voltage, and "y" could represent random noise at some instant in time. The sum "w" represents a signal-plus-noise voltage available at Receiver.

The probability distribution function of a random variable "w" is given by

$$F_w(w) = P(W \le w)$$

$$= P(x+y \le w) \quad \text{--- (1)}$$

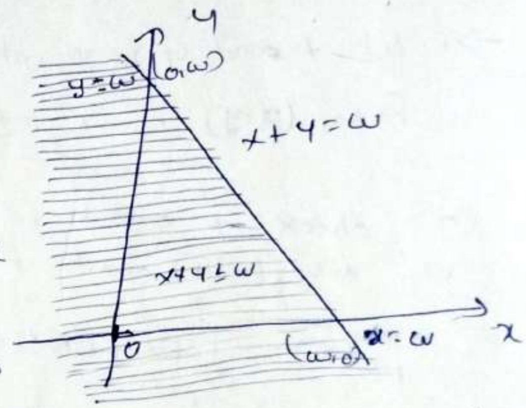


fig:- The region on xy plane where $x+y \le w$.

→ The probability corresponding to an elemental area dxdy in the xy plane located at the point (x,y) is $f_{x,y}(x,y)\,dxdy$. If we sum all such probabilities over the region where $x+y \le w$ we will obtain $F_w(w)$. Then

$$F_w(w) = P\{x+y \le w\}.$$

$$F_w(w) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{w-y} f_{x,y}(x,y)\,dx\,dy.$$

Here x and y are two independent R.V.s then $f_{x,y}(x,y) = f_x(x) \cdot f_y(y)$

$$\therefore F_w(w) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{w-y} f_x(x) \cdot f_y(y)\,dx\,dy$$

$$F_w(w) = \int_{y=-\infty}^{\infty} f_y(y) \left[\int_{x=-\infty}^{w-y} f_x(x)\,dx \right] dy \quad \text{--- (2)}$$

differentiating on both sides w.r.t w by using Leibnitz's rule

$$f_w(w) = \int_{y=-\infty}^{\infty} f_y(y) \left[\frac{d}{dw} \int_{x=-\infty}^{w-y} f_x(x)\,dx \right] dy \,-$$

$$\boxed{f_W(w) = \int_{y=-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy}$$ (8)

→ This expression is recognized as a convolution integral

$$\boxed{f_W(w) = f_Y(y) * f_X(x)} \quad (9)$$

→ The density function of the sum of two statistically independent R.Vs is equals to the convolution of their individual density function.

consider 'N' statistically independent random variables $x_1, x_2 - x_N$.
Then the sum of the random variables is given as

$$Y = x_1 + x_2 + x_3 + \cdots + x_N \quad (10)$$

now the probability density function $f_Y(y)$ is given by

$$f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) \cdots f_{X_{N-1}}(x_{N-1}) * f_{X_N}(x_N).$$

The density function of the sum of N-statistically independent random variables is equal to the convolution of their individual density functions

Note :- From convolution property $\boxed{F_Y(y) * f_X(x) = f_X(x) * f_Y(y)}$.

Problem. statistically independent random variables X & Y have respective density functions $f_X(x) = 5u(x)e^{-5x}$ and $f_Y(y) = 2u(y)e^{-2y}$. Find the density of the sum $W = X+Y$.

sol:- Given $f_X(x) = 5u(x)e^{-5x}$ $f_Y(y) = 2u(y)e^{-2y}$

we know that $F_W(w) = f_Y(y) * f_X(x)$

$$= \int_{y=-\infty}^{\infty} f_Y(y)\, f_X(w-y)\, dy.$$

$$\Rightarrow F_W(w) = \int_{-\infty}^{\infty} 2u(y)e^{-2y}\, 5u(w-y)e^{-5(w-y)}\, dy.$$

$$= 10\int_{-\infty}^{\infty} u(y)\, u(w-y)\, e^{-2y} e^{-5w} e^{5y}\, dy$$

$$= 10\, e^{-5w} \int_{-\infty}^{\infty} u(y)\, u(w-y)\, e^{3y}\, dy.$$

$$u(y) = \begin{cases} 1 & \text{for } y \geq 0 \\ 0 & \text{for } y \leq 0 \end{cases} \qquad u(w-y) = \begin{cases} 1 & w-y>0 \text{ or } y<w \\ 0 & \text{for } w-y<0 \text{ or else} \end{cases}$$

$$u(y)\, u(w-y) = \begin{cases} 1 & \text{for } 0<y<w \\ 0 & \text{else} \end{cases}$$

$$F_w(w) = 10e^{-5w} \int_0^w 1 \cdot e^{3y}\, dy = 10e^{-5w} \cdot \frac{e^{3y}}{3}\Big|_0^w$$

$$= \frac{10}{3} e^{-5w}\left[e^{3w}-1\right] \quad w \geq 0 = \frac{10}{3} e^{-2w} - \frac{10}{3} e^{-5w}$$

$$= \frac{10}{3}\left[e^{-2w} - e^{-5w}\right] \quad w>0$$

$$= \frac{10}{3}\left[e^{-2w} - e^{-5w}\right] u(w) \qquad f_w(w) = \begin{cases} \frac{10}{3}(e^{-2w} - e^{-5w}) & \text{for } w>0 \\ 0 & \text{else} \end{cases}$$

→ Find the density function of $w = x+y$ where the densities of x and y are assumed to be $f_x(x) = [u(x) - u(x-1)]$, $f_y(y) = [u(y) - u(y-1)]$

sol: given $f_x(x) = u(x) - u(x-1)$

$f_y(y) = u(y) - u(y-1)$

we know that $F_w(w) = f_x(x) * f_y(y) = \int_{-\infty}^{\infty} f_y(y) f_x(w-y)\, dy.$

$$F_w(w) = \int_{-\infty}^{\infty} [u(y) - u(y-1)]\,(u(w-y) - u(w-y-1))\, dy.$$

$$= \int_{-\infty}^{\infty} \big(u(y)\, u(w-y) - u(y-1)\, u(w-y) - u(y)\, u(w-y-1) + u(w-y-1)\, u(y-1)\big)\, dy.$$

$$F_w(w) = \int_{-\infty}^{\infty} u(y)\, u(w-y)\, dy - \int_{-\infty}^{\infty} u(y-1)\, u(w-y)\, dy - \int_{-\infty}^{\infty} u(y)\, u(w-y-1)\, dy + \int_{-\infty}^{\infty} u(w-y-1)\, u(y-1)\, dy.$$

$$u(y) \begin{cases} 1 & \text{for } y>0 \\ 0 & \text{else} \end{cases} \qquad u(w-y) = \begin{cases} 1 & \text{for } w-y>0 \Rightarrow y<w \\ 0 & \text{else} \end{cases}$$

$$u(y-1) = \begin{cases} 1 & \text{for } y>1 \\ 0 & \text{else} \end{cases} \qquad u(w-y-1) = \begin{cases} 1 & \text{for } w-y-1>0,\ y<w-1 \\ 0 & \text{else.} \end{cases}$$

$$F_w(w) = \int_0^w dy - \int_1^w dy - \int_0^{w-1} dy + \int_1^{w-1} dy.$$

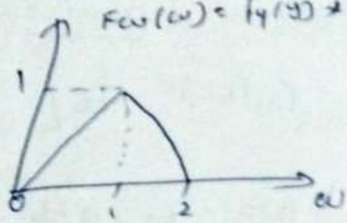
case (i): $0 \le \omega \le 1 \Rightarrow f_\omega(\omega) = \int_0^\omega 1 \cdot dy = \omega$.

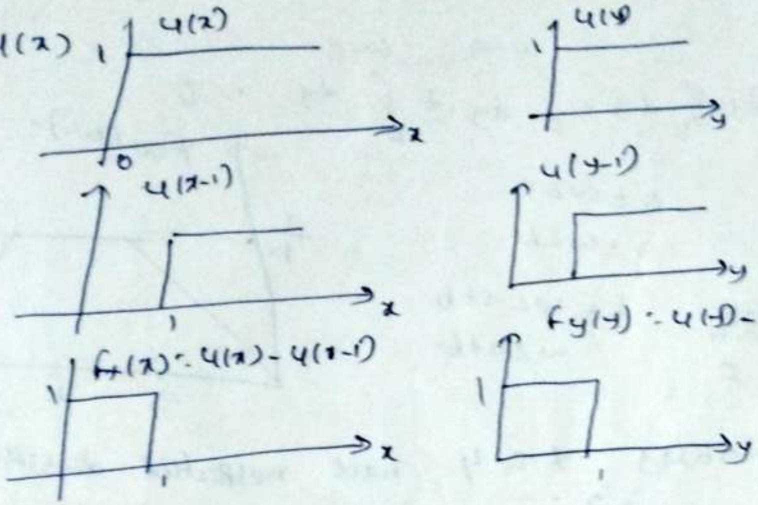
case (ii): $1 \le \omega < 2$. $f_\omega(\omega) = \int_0^\omega 1 \cdot dy - \int_0^{\omega-1} - \int_1^\omega 1\,dy$

$$= \omega - (\omega - 1) - (\omega - 1) = 2 - \omega$$

case (iii) $\omega > 2$ $\quad f_\omega(\omega) = \int_0^\omega dy - \int_1^\omega dy - \int_0^{\omega-1} dy + \int_1^{\omega-1} dy = 0.$

$$f_\omega(\omega) = \begin{cases} \omega & \text{for } 0 \le \omega < 1 \\ 2-\omega & \text{for } 1 \le \omega < 2 \\ 0 & \text{,, } \omega > 2 \end{cases}$$





NB: The response for the density function of ω is shown in fig.

→ statistically independent R.Vs X and Y have respective densities $f_X(x) = \frac{1}{a}[u(x) - u(x-a)]$, $f_Y(y) = \frac{1}{b}[u(y) - u(y-b)]$ where $0 < a < b$. Find the density function of $\omega = X + Y$.

sol: we know that $F_\omega(\omega) = f_Y(y) * f_X(x)$.

$$F_\omega(\omega) = \int_{-\infty}^{\infty} f_Y(y) f_X(\omega - y)\,dy.$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{b} u(y) - u(y-b)\right] \frac{1}{a}\left[u(\omega - y) - u(\omega - y - a)\right] dy.$$

$$= \frac{1}{ab}\left\{\int_{-\infty}^{\infty} u(y)\,u(\omega-y)\,dy - \int_{-\infty}^{\infty} u(\omega-y)\,u(y-b) - \int_{-\infty}^{\infty} u(y)\,u(\omega-y-a)\,dy \right.$$

$$+ \int_{-\infty}^{\infty} u(\omega-y-a)\,u(y-b)\,dy.$$

$$= \frac{1}{ab}\left[\int_0^\omega 1\,dy - \int_b^\omega 1 \cdot dy - \int_0^{\omega-a} 1\,dy + \int_b^{\omega-a} 1 \cdot dy.\right.$$

case i): $0 \le \omega < a$

$$F_W(\omega) = \frac{1}{ab}\int_0^{\omega} dy + 0 + 0 + 0 = \frac{\omega}{ab}$$

case ii): $a \le \omega < b$

$$F_W(\omega) = \frac{1}{ab}\left[\int_0^{\omega} 1.\,dy - \int_0^{\omega-a} dy + 0 + 0\right] = \frac{1}{ab}\left[\omega - (\omega-a)\right] = \frac{1}{b}.$$

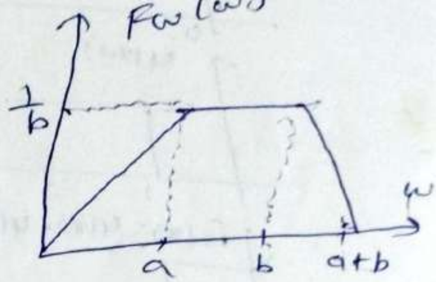
case iii): $b \le \omega < a+b$

$$F_W(\omega) = \frac{1}{ab}\left[\int_0^{\omega} dy - \int_0^{\omega-a} dy - \int_b^{\omega} dy\right] = \frac{1}{ab}\left[\omega-\omega+a\omega+b\right] = \frac{a+b-\omega}{ab}$$

case iv): $\omega \ge a+b$

$$F_W(\omega) = \frac{1}{ab}\left[\int_0^{\omega} dy - \int_b^{\omega} dy - \int_0^{\omega-a} dy + \int_b^{\omega-a} dy\right] = 0$$

$$F_W(\omega) = \begin{cases} \frac{\omega}{ab} & 0 \le \omega < a \\ \frac{1}{b} & a \le \omega < b \\ \frac{a+b-\omega}{ab} & b \le \omega < a+b \\ 0 & \omega \ge a+b. \end{cases}$$



→ The random variables X & Y have respective densities $f_X(x) = \frac{1}{a}\left[u(x) - u(x-a)\right]$ and $f_Y(y) = b\,u(y)\,e^{-by}$ where $a>0$ and $b>0$. Find and sketch the density function of $W = X+Y$, if X & Y are statistically independent.

sol: Given $f_X(x) = \frac{1}{a}\left[u(x) - u(x-a)\right]$ $f_Y(y) = b\,u(y)\,e^{-by}$.

w.k.t $f_W(\omega) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(\omega-x)\,dx$

$$f_W(\omega) = \int_{-\infty}^{\infty} \frac{1}{a}\{u(x) - u(x-a)\}\; b\cdot u(\omega-x)\, e^{-b(\omega-x)}\,dx$$

$$= \frac{b}{a}\int_{-\infty}^{\infty} u(x)\,u(\omega-x)\,e^{-b\omega+bx}\,dx - \frac{b}{a}\int_{-\infty}^{\infty} u(x-a)\,u(\omega-x)\,e^{-b\omega}\,e^{bx}\,dx$$

$$= \frac{b}{a}e^{-b\omega}\int_{-\infty}^{\infty} u(x)\,u(\omega-x)\,e^{bx}\,dx - \frac{b}{a}e^{-b\omega}\int_{-\infty}^{\infty} u(x-a)\cdot u(\omega-x)\,e^{bx}\,dx$$

$$= \frac{b}{a}e^{-b\omega}\int_0^{\omega} 1.\,e^{bx}\,dx - \frac{b}{a}e^{-b\omega}\int_a^{\omega} e^{bx}\,dx.$$

case (i): $0 < \omega \le a$

$$F_W(\omega) = \frac{b}{a}e^{-b\omega}\int_0^{\omega} e^{bx}\,dx = \frac{b}{a}e^{-b\omega}\left[\frac{e^{bx}}{b}\right]_0^{\omega}$$

$$= \frac{b}{a} e^{-bw} \left[\frac{e^{bw}}{b} - \frac{1}{b} \right] = \frac{1}{a} - \frac{1}{a} e^{-bw} = \frac{1}{a} \left[1 - e^{-bw} \right].$$ (18)

<u>case ii</u>: $w \geq a$

$$F_W(w) = \frac{b}{a} e^{-bw} \int_0^w 1 \cdot e^{bx} dx - \frac{b}{a} e^{-bw} \int_a^w e^{bx} dx.$$

$$= \frac{1}{a} \left[1 - e^{-bw} \right] - \frac{b}{a} e^{-bw} \left[\frac{e^{bx}}{b} \right]_a^w$$

$$= \frac{1}{a} \left[1 - e^{-bw} \right] - \frac{b}{a} \cdot e^{-bw} \left[\frac{e^{bw}}{b} - \frac{e^{ab}}{b} \right] = \frac{1}{a}\left[1 - e^{-bw}\right] - \frac{1}{a} - \frac{1}{a} e^{-bw+ab}$$

$$= - \frac{e^{-bw}}{a} + \frac{e^{ab} \cdot e^{-bw}}{a} = \frac{e^{-bw}}{a} \left[e^{ab} - 1 \right]$$

<u>case (iii)</u> $w \leq 0 \Rightarrow F_W(w) = 0$

$$\therefore F_W(w) = \begin{cases} 0 & w \leq 0 \\ \frac{1}{a}\left(1 - e^{-bw}\right) & \text{for } 0 < w \leq a \\ \frac{e^{-bw}}{a}\left(e^{ab} - 1\right) & \text{for } w > a \end{cases}$$

<u>NOTE</u>: The characteristic function of a normalized Gaussian R.V (mean is "0", variance = 1) is given by $\phi(w) = \exp(-w^2/2)$.

<u>CENTRAL LIMIT theorem</u>:-

central limit theorem states that the probability distribution function of the sum of a large number of independent random variables approaches a gaussian distribution.

<u>Unequal distributions</u>:-

Let $\bar{X}_i$, $\sigma_{X_i}^2$ be the means and variances of N-random variables $X_1, X_2, \ldots X_N$ respectively.

The central limit theorem states the sum $Y_N = X_1 + X_2 + \ldots + X_N$ which has mean $\bar{Y}_N = \bar{X}_1 + \bar{X}_2 + \ldots + \bar{X}_N$ and variance $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \ldots + \sigma_{X_N}^2$, has probability distribution that approaches gaussian as $N \to \infty$.

→ The necessary conditions for this theorem are difficult to state but sufficient conditions are known to be

$$\sigma_{X_i}^2 > B_1 > 0 \quad i = 1, 2 \ldots N$$

and $E\left[\,|x_i - \bar{x}_i|^3\right] > D_2 \quad i = 1, 2, \cdots N$

Here D_1 and D_2 are the +ve numbers.

Equal distributions:-

$x_1, x_2 \cdots x_N$ are "N" statistically independent random variables and assume they have equal distributions.

$$\bar{x}_i = \bar{x} \quad \text{for } i = 1, 2, \cdots N$$

$$\sigma_{x_i}^2 = \sigma_x^2, \quad \text{for } i = 1, 2 \cdots N$$

Because all ~~~~ values of "i" x_i has equal distribution

$$\therefore \quad y_N = x_1 + x_N + \cdots + x_N$$

$$\bar{y}_N = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_N$$

$$\Rightarrow \quad \sigma_{y_N}^2 = N\sigma_x^2$$

$$\Rightarrow \quad \sigma_{y_N} = \sqrt{N}\,\sigma_x$$

as $N \to \infty$ the distribution of y_N is Gaussian according to central limit theorem. Let $w_N = \dfrac{y_N - \bar{y}_N}{\sigma_{y_N}}$

If we prove that w_N is normalized Gaussian then y_N is also Gaussian R.V with mean $\bar{y}_N$ and variance is σ_{y_N}.

$$\Rightarrow w_N = \frac{(x_1 + x_2 + \cdots + x_N) - (\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_N)}{\sqrt{N}\,\sigma_x}$$

$$w_N = \frac{\sum_{i=1}^{N}(x_i - \bar{x}_i)}{\sqrt{N}\,\sigma_x}.$$

Proof:- In order to prove the central limit theorem to show that the characteristic function of w_N is a normalized gaussian R.V (mean: 0, variance: 1) which is the $\phi_{w_N}(\omega) = \exp(-\omega^2/2)$.

we know that the characteristic function of R.V w_N is given by

$$\phi_{w_N}(\omega) = E\left[e^{j\omega w_N}\right].$$

$$\phi_{w_N}(\omega) = E\left[\exp\left(j\omega \frac{\sum_{i=1}^{N}(x_i - \bar{x}_i)}{\sqrt{N}\sigma_x}\right)\right].$$

Here $x_1, x_2 \dots x_N$ are independents and also equal distributions.

$$\phi_{w_N}(\omega) = \left[E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_x}(x_i - \bar{x}_i)\right)\right]\right]^N \quad -(1).$$

The exponential in eq (1) is expanded in a Taylor polynomial with a remainder term R_N/N as

consider

$$E\left[\exp\left(\frac{j\omega}{\sqrt{N}\sigma_x}(x_i - \bar{x}_i)\right)\right] = E\left[1 + \frac{j\omega}{\sqrt{N}\sigma_x}(x_i - \bar{x}_i) + \left(\frac{j\omega}{\sqrt{N}\sigma_x}\right)^2 \frac{(x_i - \bar{x}_i)^2}{2} + \frac{R_N}{N}\right].$$

$$= 1 - \frac{\omega^2}{2N} + E[R_N]/N$$

Here $\frac{R_N}{N}$ represents the remainder term in the $\exp\left[\frac{j\omega}{\sqrt{N}\sigma_x}(x_i - \bar{x}_i)\right]$.

$$= 1 + \frac{j\omega}{\sqrt{N}\sigma_x} E[(x_i - \bar{x}_i)] + \frac{(j\omega)^2}{2N\sigma_x^2} E[(x_i - \bar{x}_i)^2] + \frac{E[R_N]}{N}$$

$$= 1 - \frac{\omega^2}{2N\cancel{\sigma_x^2}}\cancel{\sigma_x^2} + \frac{E[R_N]}{N} = 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \quad -(2)$$

As $N \to \infty$ then $E[R_N]$ approaches to zero, sub (2) in (1)

$$\phi_{w_N}(\omega) = \left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]^N$$ apply natural logarithm on both sides

$$\log_e \phi_{w_N}(\omega) = N\log\left[1 - \frac{\omega^2}{2N} + E\left(\frac{R_N}{N}\right)\right].$$

$$= N\log\left[1 - \left(\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right)\right]$$

w.k.T $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right).$

$$\log_e \phi_{w_N}(\omega) = -N\left\{\left[\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right] + \frac{1}{2}\left[\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right]^2 + \cdots\right\}$$

$$\log_e \phi_{w_N}(\omega) = \frac{-\omega^2}{2} + E[R_N] - \frac{N}{2}\left[\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right]^2 + \cdots$$

as $N \to \infty$

$$\lim_{N\to\infty}\left\{\log_e \phi_{W_N}(\omega)\right\} = \frac{-\omega^2}{2}$$

$$\log_e\left\{\lim_{N\to\infty}\phi_{W_N}(\omega)\right\} = \frac{-\omega^2}{2}$$

$$\lim_{N\to\infty}\phi_{W_N}(\omega) = e^{-\omega^2/2}$$

characteristic function of W_N is normalized gaussian of $N\to\infty$
Hence central limit theorem is proved.

Problem:- Given the function $f_{XY}(x,y) = \begin{cases}(x^2+y^2)/8\pi & x^2+y^2<b \\ 0 & \text{else}\end{cases}$

a) Find the constant b that is a valid joint density function
b) Find $P\{0.5b < x^2+y^2 < 0.8b\}$.

Sol:- Given $f_{XY}(x,y) = \begin{cases} x^2+y^2/8\pi & x^2+y^2<b \\ 0 & \text{else}\end{cases}$

w.k.t $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

Here $x^2+y^2<b$ represents the circle of radius $\sqrt{b}$ with centre origin

convert (x,y) to a polar coordinates (r,θ).

Here $x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$ $\quad dx\,dy = r\,dr\,d\theta$

$$\Rightarrow \iint_{\text{circular plane}} \frac{x^2+y^2}{8\pi}\,dx\,dy = 1 \Rightarrow \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{b}} \frac{r^2}{8\pi}\, r\,dr\,d\theta = 1 = \frac{1}{8\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{b}} r^3\,dr\,d\theta = 1$$

$$= \frac{1}{8\pi}\int_0^{2\pi} \frac{r^4}{4}\Big|_0^{\sqrt{b}} d\theta = \int_0^{2\pi}\frac{1}{8\pi}\left[\frac{b^2}{4}\right]d\theta = \frac{b^2}{32\pi}(2\pi) = \frac{b^2}{16} = 1 \quad \boxed{b=4}$$

$$\rightarrow P\{0.5b < x^2+y^2 < 0.8b\} = P\{0.5b < r^2 < 0.8b\}.$$

$$= P\{\sqrt{0.5b} < r < \sqrt{0.8b}\}$$

$$= \int_{\theta=0}^{2\pi}\int_{\sqrt{0.5b}}^{\sqrt{0.8b}} \frac{r^3}{8\pi}\,dr\,d\theta = \frac{1}{8\pi}\int_{\theta=0}^{2\pi} \frac{r^4}{4}\Big|_{\sqrt{0.5b}}^{\sqrt{0.8b}} d\theta = \frac{1}{8\pi}\int_0^{2\pi}\left[\frac{(0.8b)^2-(0.5b)^2}{4}\right]d\theta$$

$$= \frac{1}{8\pi}\int_0^{2\pi}\left(\frac{(3.2)^2-(2)^2}{4}\right)d\theta = \frac{1}{8\pi}\int_{\theta=0}^{2\pi}\frac{6.24}{4}\,d\theta =$$

$$= \frac{0.195\cdot(2\pi)}{\pi} = 0.39.$$

Operations on multiple Random variables :-

Expected value of a function of Random variable :-

If $g(x,y)$ is a function of two random variables x & y then the $E[g(x,y)]$ is given by
$$E[g(x,y)] = \bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{xy}(x,y)\, dx\,dy \quad -(1)$$

For discrete random variables
$$E[g(x,y)] = \bar{g} = \sum_{i=1}^{M}\sum_{j=1}^{N} g(x_i, y_j)\, P(x_i, y_j) \quad -(2)$$

→ consider N-random variables $x_1, x_2, x_3 \cdots x_N$ with the function $g(x_1, x_2, x_3 \cdots x_N)$ of N-random variables then the Expected value of N-R.V function $g(x_1, x_2, \cdots x_N)$ is given by
$$E[g(x_1, x_2, x_3 \cdots x_N)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1, x_2 \cdots x_N)\, f_{x_1, x_2 \cdots x_N}(x_1, x_2 \cdots x_N)\, dx_1\, dx_2 \cdots dx_N \quad -(3)$$

→ Find the expected value of a sum of N-weighted R.Vs (or) show that the mean value of a weighted sum of random variables equals to the weighted sum of mean values.

sol:- let $g(x_1, x_2, \cdots x_N) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N$
$$= \sum_{i=1}^{N} \alpha_i x_i$$

$$E\left[\sum_{i=1}^{N} \alpha_i x_i\right] = \sum_{i=1}^{N} E[\alpha_i x_i]$$
$$= \sum_{i=1}^{N} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (\alpha_i x_i)\, f_{x_1, x_2, \cdots x_N}(x_1, x_2 \cdots x_N)\, dx_1\, dx_2 \cdots dx_N$$
$$= \sum_{i=1}^{N} \int_{-\infty}^{\infty} \alpha_i x_i\, f_{x_i}(x_i)\, dx_i$$

$$E\left[\sum_{i=1}^{N} \alpha_i x_i\right] = \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} x_i\, f_{x_i}(x_i)\, dx_i$$
$$= \sum_{i=1}^{N} \alpha_i E[x_i].$$ Therefore we conclude that the mean value of weighted sum of random variables is equals to the weighted sum of mean values.

→ Random variables X & Y have the density function

$f_{XY}(x,y) = \begin{cases} \frac{1}{24} & 0<x<6 \text{ and } 0<y<4 \\ 0 & \text{else} \end{cases}$ Find the expected value of the function.

$g(x,y) = (xy)^2$

<u>sol</u>: $E[g(x,y)] = \int_{y=0}^{4}\int_{x=0}^{6}(xy)^2 f_{XY}(x,y)\,dx\,dy = \int_{y=0}^{4}\int_{x=0}^{6}(xy)^2 \cdot \frac{1}{24}\,dx\,dy.$

$= \frac{1}{24}\int_{y=0}^{4} y^2\left(\int_{x=0}^{6} x^2\,dx\right)dy = \frac{1}{24}\int_{y=0}^{4} y^2 \cdot \frac{x^3}{3}\Big|_0^6$

$= \frac{1}{24} \times 3 \left[\frac{6\times6\times6}{3}\right]\left[\frac{y^3}{3}\right]_0^4 = 3 \times \left[\frac{64}{3}\right] = 64.$

→ Three statistically independent random variables X_1, X_2 & X_3 have mean values $\bar{X}_1 = 3$, $\bar{X}_2 = 6$, $\bar{X}_3 = -2$. Find the mean value of the following (a) $g(X_1,X_2,X_3) = X_1 + 3X_2 + 4X_3$.

b) $g(X_1,X_2,X_3) = X_1X_2X_3$ c) $g(X_1,X_2,X_3) = -2X_1X_2 - 3X_1X_3 + 4X_2X_3$

d) $g(X_1,X_2,X_3) = X_1 + X_2 + X_3$.

<u>sol</u> (a) $E[g(X_1,X_2,X_3)] = E[X_1 + 3X_2 + 4X_3]$

$= E[X_1] + 3E[X_2] + 4E[X_3]$

$= 3 + 3(6) + 4(-2) = 13.$

b) $E[g(X_1,X_2,X_3)] = E[X_1X_2X_3]$

Given X_1, X_2, X_3 are statistically independent

$= E[X_1]E[X_2]E[X_3] = \bar{X}_1\bar{X}_2\bar{X}_3 = -36.$

c) $E[g(X_1,X_2,X_3)] = E[-2X_1X_2 - 3X_1X_3 + 4X_2X_3].$

$= -2E[X_1]E[X_2] - 3E[X_1]E[X_3] + 4E[X_2]E[X_3].$

$= -2(3)(6) - 3(3)(-2) + 4(6)(-2) = -16.$

d) $E[g(X_1,X_2,X_3)] = E[X_1 + X_2 + X_3]$

$= E[X_1] + E[X_2] + E[X_3]$

$= 3 + 6 - 2 = 7$

→ Find the mean value of the function $g(x,y) = x^2+y^2$ where (51)
X & Y are R.Vs defined by the density function $f_{XY}(x,y) = \dfrac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}$

sol:- $E[g(x,y)] = E[x^2+y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)\, f_{XY}(x,y)\,dx\,dy.$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)\,\frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}\,dx\,dy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2\,\frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2\,\frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}\,dx\,dy.$$

$$= \int_{-\infty}^{\infty} x^2\,\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx \int_{-\infty}^{\infty} \frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dy + \int_{-\infty}^{\infty} y^2\,\frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dy \int_{-\infty}^{\infty} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx$$

W.K.T $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx = \int_{-\infty}^{\infty} \frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dy = 1$

$$E[x^2+y^2] = \int_{-\infty}^{\infty} x^2\,\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx + \int_{-\infty}^{\infty} y^2\,\frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dy.$$

consider $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx$ let $t = \dfrac{x}{\sqrt{2}\sigma} \Rightarrow x = (\sqrt{2}\,t)\sigma$

$x = -\infty, \Rightarrow t = -\infty$
$x = \infty \quad t = \infty$

$$\rightarrow \int_{-\infty}^{\infty} \frac{2\sigma^2 t^2}{\sqrt{2\pi\sigma^2}}\, e^{-t^2}\sqrt{2}\,\sigma\, dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\left(2\int_0^{\infty} t^2 e^{-t^2}\,dt\right) = \frac{4\sigma^2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{4} = \sigma^2$$

similarly $\displaystyle\int_{-\infty}^{\infty} \frac{y^2 e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dy = \sigma^2$

$E[x^2+y^2] = \sigma^2+\sigma^2 = 2\sigma^2.$

→ Two R.Vs have a uniform density on a circular region defined by

$f_{XY}(x,y) = \begin{cases} \dfrac{1}{\pi r^2} & x^2+y^2 \le r^2 \\ 0 & \text{else} \end{cases}$ find the mean value of the function $g(x,y) = x^2+y^2.$

sol:- Given $x^2+y^2 \le r^2$ represents the circle of radius r and centre at origin.

Now convert (x,y) into polar co-ordinates $x = r\cos\theta$ $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}(y/x)$ and $dx\,dy = r\,dr\,d\theta$

$$E\{x^2+y^2\} = \int_{\theta=0}^{2\pi}\int_{r=0}^{r} r^2 \cdot \frac{1}{\pi r^2}\, r\, dr\, d\theta = \frac{1}{\pi}\frac{r^2}{2}\Big|_0^{r}\ \theta\Big|_0^{2\pi} = \underline{r^2}.$$

→ The density function of two random variables X & Y is $f_{XY}(x,y) = u(x)u(y)e^{-4(x+y)}$ find the mean value of the function $g(x,y) = e^{-2(x^2+y^2)}$.

sol: Given $g(x,y) = e^{-2(x^2+y^2)}$

$f_{XY}(x,y) = u(x)u(y)e^{-4(x+y)}$.

mean value $E\{e^{-2(x^2+y^2)}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2(x^2+y^2)}\, u(x)u(y)\, e^{-4(x+y)}\, dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2x^2} e^{-2y^2} u(x)u(y)\, e^{-4x} e^{-4y}\, dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \left\{e^{-2x^2} e^{-4x}\,dx\right\} e^{-2y^2} u(y)\, e^{-4y}\, dy$$

$$= \int_{-\infty}^{\infty} e^{-2y^2} u(y)\, e^{-4y}\, dy \int_{0}^{\infty} e^{-2(x^2+2x)}\, dx.$$

Joint Moments about the origin:-

Joint moments about the origin is denoted by m_{nk}.

Let X & Y are two R.Vs with joint density function $f_{XY}(x,y)$ then moments about the origin is defined as

$$m_{nk} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\,dx\,dy \quad -(1)$$

$m_{n0} = E\{X^n\}$ = moment of "X"

$m_{0k} = E\{Y^k\}$ = moment of "Y".

The order of the joint moment is $n+k$.

Zeroth order Moment:- $m_{00} = E\{X^0 Y^0\} = E(1) = 1$.

1st order moments:- m_{10}, m_{01} are first order moments.

$m_{10} = E[X^1] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \bar{X}$ = moment of X

$m_{01} = E\{Y^1\} = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \bar{Y}$ = moment of Y

The first order moments m_{01} and m_{10} are the excepted values of X & Y.

2nd order moments:-

m_{20}, m_{02}, and m_{11} are second order moments of X and Y.

$m_{20} = E[X^2]$ = Mean square value of X

$m_{02} = E[Y^2]$ = " " " Y

$m_{11} = E[XY]$ = correlation.

The second order moment $m_{11} = E\{XY\}$ is called the correlation of X and Y, denoted by R_{XY}.

$$R_{XY} = m_{11} = E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy.$$

→ If X and Y are statistically independent then X & Y are said to be uncorrelated i.e $R_{XY} = E[X]E(Y)$.

→ If X and Y are said to be orthogonal then $R_{XY} = 0$.

Joint central moments:-

Joint central moments of two Random variables can be denoted by μ_{mk}.
If X and Y are two random variables with joint density function $f_{XY}(x,y)$ then the central moments μ_{mk} is given by

$$\mu_{mk} = E[(X-\bar{X})^m (Y-\bar{Y})^k]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})^m (y-\bar{Y})^k f_{X,Y}(x,y)\,dx\,dy$$

Zeroth central moment $\mu_{00} = E[(X-\bar{X})^0 (Y-\bar{Y})^0] = E(1) = 1$

First order joint central moment

μ_{01}, μ_{10} are the first order joint central moments i.e

$$\mu_{10} = E[X-\bar{X}]^1 = E[X-\bar{X}] = 0.$$

$$\mu_{01} = E[(X-\bar{X})^0 (Y-\bar{Y})^1] = E[Y-\bar{Y}] = 0.$$

Second order central moments

$\mu_{20}, \mu_{02}, \mu_{11}$ are the second order central moments.

consider $\mu_{20} = E[(X-\bar{X})^2 (Y-\bar{Y})^0] = E[(X-\bar{X})^2]$.

$$= E[X^2 + \bar{X}^2 - 2X\bar{X}] = E[X^2] + \bar{X}^2 - 2\bar{X}^2$$

$$= E[X^2] - \bar{X}^2 = var(X) = \sigma_X^2$$

Here μ_{20} and μ_{02} represents the variance of X and Y.

$$\mu_{02} = E[(X-\bar{X})^0 (Y-\bar{Y})^2] = E[(Y-\bar{Y})^2] = E(Y^2) - \bar{Y}^2$$

$$= var(Y) = \sigma_Y^2$$

variance of X and Y.

→ The 2nd order central moment μ_{11} is called co-variance of X and Y.
i.e It is represented with C_{XY}.

$$C_{XY} = \mu_{11} = E[(X-\bar{X})(Y-\bar{Y})] = E[XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}].$$

$$= E[XY] - E[X]\bar{Y} - \bar{X}E[Y] + \bar{X}\bar{Y}$$

$$= E[XY] - \bar{X}\bar{Y}$$

$$\boxed{C_{XY} = \mu_{11} = E[XY] - \bar{X}\bar{Y}}$$ It is called co-variance.

$$\therefore C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[X]E[Y].$$

case i) If X & Y are statistically independent (or) X and Y are uncorrelated then $C_{XY} = \mu_{11} = 0 \Rightarrow \boxed{C_{XY} = 0}$

Coll (ii) If x & y are orthogonal then $R_{xy} = 0$ and C_{xy} becomes

$$\Rightarrow \boxed{\mu_{11} = -\bar{x}\bar{y}}$$

co-relation co-efficient (ρ)

The normalized second order central moment is called co-relation coefficient. It is denoted with ρ.

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\mu_{02}}} = \frac{C_{xy}}{\sigma_x \sigma_y} \quad -(5)$$

$$\Rightarrow \rho = \frac{C_{xy}}{\sigma_x\sigma_y} = \frac{E[(x-\bar{x})(y-\bar{y})]}{\sigma_x \sigma_y}.$$

Properties:-

→ corelation coefficient ρ always lies b/w -1 to 1

→ If x & y are independent then $\rho = 0$ [$\because C_{xy} = 0$]

→ If $x = y$ then $\rho = 1$

Properties of co-variance:-

→ If x & y are two random variables then $C_{xy} = R_{xy} - \bar{x}\bar{y}$

Proof:- we know that $C_{xy} = \mu_{11} = E[(x-\bar{x})(y-\bar{y})]$.

$$\mu_{11} = E[xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y}] = E\{xy\} - \bar{x}\bar{y} \cancel{- \bar{x}\bar{y} + \bar{x}\bar{y}})$$

$$= E[xy] - \bar{x}\bar{y} = R_{xy} - \bar{x}\bar{y}$$

→ If two R.Vs x and y are independent then co-variance $C_{xy} = 0$.

w.k.T $C_{xy} = R_{xy} - \bar{x}\bar{y} = E[xy] - \bar{x}\bar{y} = E(x)E(y) - \bar{x}\bar{y}$

$$= \bar{x}\bar{y} - \bar{x}\bar{y} = 0$$

→ If x and y are two random variables then co-variance at

$$Cov(ax, by) = ab\,Cov(x, y) = ab\,C_{xy}.$$

Proof:- w.k.T $Cov(x,y) = C_{xy} = E[(x-\bar{x})(y-\bar{y})]$

$$\Rightarrow Cov(ax+by) = E[(ax - \bar{a}x)(by - \bar{b}y)].$$

$$= E[a[x-\bar{x}]\,b(y-\bar{y})] = abE[(x-\bar{x})(y-\bar{y})]$$

$$= ab\,Cov(x,y) = ab\,C_{xy}.$$

→ If x and y are two random variables then variance of

$$x+y = Var(x) + Var(y) + 2C_{xy}.$$

similarly $Var(x-y) = Var(x) + Var(y) - 2C_{xy}$.

Proof: we know that $\mathrm{var}(x) = E[x^2] - \bar{x}^2$

$$\mathrm{var}(x+y) = E\left[(x+y)^2 - (\overline{x+y})^2\right]$$

$$= E\left[x^2+y^2+2xy\right] - (\bar{x}+\bar{y})^2$$

$$= E[x^2] + E[y^2] + 2E[xy] - \bar{x}^2 - \bar{y}^2 - 2\bar{x}\bar{y}$$

$$= E[x^2] - \bar{x}^2 + E[y^2] - \bar{y}^2 + 2\left[E[xy] - \bar{x}\bar{y}\right].$$

$$= \mathrm{var}(x) + \mathrm{var}(y) + 2C_{xy}$$

similarly $\mathrm{var}(x-y) = \mathrm{var}(x) + \mathrm{var}(y) - 2C_{xy}$.

Problem:- Random variables x & y have the joint density function

$$f_{xy}(x,y) = \begin{cases} \dfrac{(x+y)^2}{40} & -1<x<1 \text{ and } -3<y<3 \\ 0 & \text{else} \end{cases}$$

a) find all the second order moments of x and y

b) what are the variances of x & y

c) what is the correlation coefficient -

sdu. we know that $m_{nk} = E[x^n y^k] = \iint x^n y^k f_{xy}(x,y)\,dx\,dy$.

$$= \int_{-3}^{3}\int_{-1}^{1} x^n y^k \frac{(x+y)^2}{40}\,dx\,dy = \frac{1}{40}\int_{-3}^{3}\int_{-1}^{1} x^n y^k (x^2+y^2+2xy)\,dx\,dy.$$

$$= \frac{1}{40}\int_{-3}^{3}\int_{-1}^{1} x^{n+2}y^k + x^n y^{k+2} + 2x^{n+1}y^{k+1}\,dx\,dy.$$

$$= \frac{1}{40}\int_{-3}^{3} \left. y^k \frac{x^{n+3}}{n+3} + y^{k+2}\frac{x^{n+1}}{n+1} + 2\cdot\frac{x^{n+2}}{n+2}y^{k+1}\right|_{-1}^{1} dy$$

$$= \frac{1}{40}\int_{-3}^{3}\left[y^k\frac{1}{n+3} + y^{k+2}\frac{1}{n+1} + \frac{2}{n+2}y^{k+1} - \frac{y^k(-1)^{n+3}}{n+3} - \frac{y^{k+2}(-1)^{n+1}}{n+1} - \frac{2}{n+2}(-1)^{n+2}y^{k+1}\right]dy$$

$$= \frac{1}{40}\int_{-3}^{3} y^k\left[\frac{1}{n+3} - \frac{(-1)^{n+3}}{n+3}\right] + y^{k+2}\left[\frac{1}{n+1} - \frac{(-1)^{n+1}}{n+1}\right] + 2y^{k+1}\left[\frac{1}{n+2} - \frac{(-1)^{n+2}}{n+2}\right]dy$$

$$= \frac{1}{40}\int_{-3}^{3} \frac{y^k}{n+3}\left[1-(-1)^{n+3}\right] + \frac{y^{k+2}}{n+1}\left(1-(-1)^{n+1}\right) + \frac{2y^{k+1}}{n+2}\left(1-(-1)^{n+2}\right)dy$$

$$= \frac{1}{40}\left[\frac{1-(-1)^{n+3}}{n+3}\frac{y^{k+1}}{k+1} + \frac{1-(-1)^{n+1}}{n+1}\frac{y^{k+3}}{k+3} + \frac{2\left(1-(-1)^{n+2}\right)}{n+2}\frac{y^{k+2}}{k+2}\right]_{-3}^{3}$$

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$$= \frac{1}{40}\left\{ \frac{1-(-1)^{m+3}}{m+3}\cdot\frac{3^{k+1}}{k+1} + \frac{1-(-1)^{m+1}}{m+1}\frac{3^{k+3}}{k+3} + 2\left(\frac{1-(-1)^{m+2}}{m+2}\frac{3^{k+2}}{k+2}\right.\right.$$

$$\left. - \frac{(1-(-1)^{m+3}}{m+3}\frac{(-3)^{k+1}}{k+1} - \frac{(1-(-1)^{m+1}}{m+1}\frac{(-3)^{k+3}}{k+3} - 2\frac{(1-(-1)^{m+2}}{m+2}\frac{(-3)^{k+2}}{k+2}\right\}$$

$$m_{mk} = \frac{1}{40}\;\frac{1-(-1)^{m+3}}{(m+3)(k+1)}\left[3^{k+1}-(-3)^{k+1}\right) + \frac{1-(-1)^{m+1}}{(m+1)(k+3)}\left[3^{k+3}-(-3)^{k+3}\right)$$

$$+ 2\left(\frac{1-(-1)^{m+2}}{(m+2)(k+2)}\left[3^{k+2}-(-3)^{k+2}\right)\right.$$

$m_{20} = m=2,\ k=0$

$$m_{20} = \frac{1}{40}\left[\frac{1-(-1)^5}{(3+2)\cdot 1}\left(3^{1}-(-3)^{1}\right) + \frac{1-(-1)^3}{(2+1)(3)}\left(3^3-(-3)^3\right)\right.$$

$$+ 2\left[\frac{1-(-1)^4}{(4)(2)}\left[3^4-(-3)^4\right]\right] =$$

$$= \frac{1}{40}\left[\frac{1+1}{7}\times 6 + \frac{1+1}{6}(54)\right) = \frac{1}{40}\left[\frac{12}{7}+18\right]$$

$$= \frac{1}{40}\left[\frac{12+126}{7}\right] \cancel{126}$$

$$= \frac{138}{280} = 0.36$$

$$\to m_{02} = \frac{1}{40}\left[\frac{1-(-1)^3}{3(3)}\left[27+27\right) + \frac{1-(-1)^3}{5}\left[3^5-(-3)^5\right) + \right.$$

$$2\left(\frac{1-(-1)^2}{4}\left[3^4-3^4\right) = \frac{1}{40}\left[\frac{2}{9}\times 2(27)\right] + \frac{2}{5}\left[2\cdot 3^5\right] + $$

$$= \frac{1}{40}\left[4\times 3 + \frac{4}{5}\times 3^5\right) = \frac{1}{40}\left[12 + 4\frac{\times 3^5}{5}\right]$$

$$= \underline{5.16}.$$

$\underline{m=1\ k=1}$

$$m_{11} = \frac{1}{40}\left[\frac{1-(-1)^4}{8}\left[3^3+3^3\right] + 1-(-1)^2(0) + 2\frac{(1-(-1)^3}{9}\cdot[54]\right.$$

$$= \frac{1}{40}\left[\frac{2+2}{9}\left[2\cdot 27\right) = \frac{24}{40} = 0.6\right.$$

$m_{11} = 0.6$

Variancy of x & y :-

· variance of $x = \sigma_x^2 = E[x^2] - \bar{x}^2 = m_{20} - m_{10}^2$

$n=1, k=0$ $\quad m_{10} = \frac{2(2)}{3(4)} [3^2 - (-3)^2] = 0$

$\sigma_x^2 = 0.36$

→ variance of $y = \sigma_y^2 = E[y^2] - \bar{y}^2 = m_{02} - m_{01}^2$

$m_{01} = n=0, k=1$

$m_{01} = \frac{1}{40} \{ \frac{2}{3(4)}(0) \} = 0$

$\sigma_y^2 = m_{02} - m_{01}^2 = 5.16$

c) w.k.t co-relation coefficient $= \frac{C_{xy}}{\sigma_x \sigma_y}$

put $n=1, y=1$ $\quad C_{xy} = E[xy] - \bar{x}\bar{y}$

$E[xy] = \frac{1}{40} [\frac{2(2)}{3(4)} (2 - 27)] = 0.6$

correlation coefficient $= \frac{0.6 - 0}{\sqrt{0.36 \times 5.16}} = 0.44.$

→ Random variables x & y have the joint density function

$$f_{xy}(x,y) = \begin{cases} \frac{2}{43}(x + 0.5y)^2 & 0 < x < 2 \text{ \& } 0 < y < 3 \\ 0 & \text{else} \end{cases}$$

Find (i) all first order and second order moments.

(ii) Find the co-variance (iii) Are x and y are co-related.

sol. $f_{xy}(x,y) = \begin{cases} \frac{2}{43}(x + 0.5y)^2 & 0 < x < 2 \text{ \& } 0 < y < 3 \\ 0 & \text{else} \end{cases}$

w.k.t $m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\, dx\, dy.$

$E[x^n y^k] = \int_0^3 \int_0^2 x^n y^k \frac{2}{43} (x + 0.5y)^2\, dx\, dy$

$= \frac{2}{43} \int_0^3 \int_0^2 x^n y^k \{ x^2 + \frac{1}{4} y^2 + xy \}\, dx\, dy$

$= \frac{2}{43} \int_0^3 \int_0^2 (x^{n+2} y^k + \frac{1}{4} x^n y^{k+2} + x^{n+1} y^{k+1})\, dx\, dy.$

$= \frac{2}{43}\int_0^3 \left[y^k \cdot \frac{x^{m+3}}{m+3}\Big|_0^2 + \frac{1}{4} y^{k+2} \frac{x^{m+1}}{m+1}\Big|_0^2 + y^{k+1} \cdot \frac{x^{m+2}}{m+2}\Big|_0^2 \right\} dy.$ (55)

$= \frac{2}{43}\int_0^3 \left\{ y^k \cdot \frac{2^{m+3}}{m+3} + \frac{1}{4} y^{k+2} \frac{2^{m+1}}{m+1} + y^{k+1} \cdot \frac{2^{m+2}}{m+2} \right. dy.$

$= \frac{2}{43}\left[\frac{2^{m+3}}{m+3} \cdot \frac{y^{k+1}}{k+1} + \frac{1}{4} \cdot \frac{2^{m+1}}{m+1} \cdot \frac{y^{k+3}}{k+3} + \frac{y^{k+2}}{k+2} \cdot \frac{2^{m+2}}{m+2} \right]\Big|_0^3$

$= \frac{2}{43}\left[\frac{2^{m+3} \cdot 3^{k+1}}{(m+3)(k+1)} + \frac{1}{4} \frac{2^{m+1} \cdot 3^{k+3}}{(m+1)(\cancel{k+3})} + \frac{3^{k+2}\, 2^{m+2}}{(m+2)(k+2)} \right\}$

$m_{01} \Rightarrow m=0,\ k=1$

$m_{01} = \frac{2}{43}\left\{ \frac{8 \cdot 3^2}{3(2)} + \frac{1}{4} \frac{2(3)^4}{1(4)} + \frac{3^3 \cdot 2^2}{2(3)} \right)$

$= \frac{2}{43}\left\{ \frac{8 \times 9}{6} + \frac{1}{4} \times \frac{2 \times 3^4}{4} + \frac{27 \times 4}{6} \right) = 1.866$

$m_{10} \Rightarrow m=1,\ k=0$

$= \frac{2}{43}\left\{ \frac{2^4 \times 3}{4} + \frac{1}{4} \cdot \frac{2^2 \cdot 3^3}{(2)(3)} + \frac{3^2\, 2^3}{(3)(2)} \right] = \frac{2}{43}\left[12 + 4.5 + 12\right)$

$= 1.3255$

II order moments :-

$m_{20} \Rightarrow m=2,\ k=0$

$m_{20} = \frac{2}{43}\left[\frac{2^5 \cdot 3}{5 \cdot 1} + \frac{1}{4} \cdot \frac{2^3 \cdot 3^3}{(3)(3)} + \frac{2^4 \cdot 3^2}{4(2)} \right) = 2.009.$

$m_{02} \Rightarrow m=0,\ k=2$

$m_{02} = \frac{2}{43}\left\{ \frac{2^3\, 3^3}{3(3)} + \frac{1}{4} \frac{2 \cdot 3^5}{(1)(5)} + \frac{3^4\, 2^2}{4(2)} \right)$

$= \frac{2}{43}\left\{ 8 \times 3 + \frac{1}{2} \frac{3^5}{5} + \frac{3^4}{2} \right) = 4.13$

$m_{11} = \frac{2}{43}\left[\frac{2^4\, 3^2}{4(2)} + \frac{1}{4} \frac{2^2 \cdot 3^4}{2(4)} + \frac{3^3\, 2^3}{9} \right] = 2.424$

b) covariance $= C_{XY} = E[XY] - \bar{X}\bar{Y} = m_{11} - m_{10}\, m_{01}$

$= 2.424 - (1.3255)(1.866) = -0.049$

c) $E(XY) \neq E(X)E(Y)$ They are not uncorreleted

$\Rightarrow$ A joint density function is $f_{XY}(x,y) = \begin{cases} x(y+1.5) & 0<x<1,\ 0<y<1 \\ 0 & \text{else} \end{cases}$

Find all the joint moments m_{mk} hee m & $k = 0,1,2 - - -$

Sol:- joint moments $m_{mk} = E[X^m Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^k f_{XY}(x,y)\, dx\, dy$

$= \int_0^1\int_0^1 x^m y^k (xy + 1.5x)\, dx\, dy = \int_0^1\int_0^1 x^{m+1} y^{k+1} + 1.5 x^{m+1} y^k)\, dx\, dy$

$= \int_0^1 y^{k+1} \frac{x^{m+2}}{m+2} + 1.5 \frac{x^{m+2}}{m+2} y^k \Big|_0^1 dy = \int_0^1 \frac{y^{k+1}}{m+2} + \frac{1.5}{m+2} y^k\, dy.$

$= \frac{y^{k+2}}{(m+2)(k+2)} + \frac{1.5\, y^{k+1}}{(m+2)(k+1)}\Big|_0^1 = \frac{1}{(m+2)(k+2)} + \frac{1.5}{(m+2)(k+1)}$

$= \frac{1}{m+2}\left[\frac{k+1+1.5(k+2)}{(k+1)(k+2)}\right].$

$m_{mk} = \frac{2.5k+4}{(m+2)(k+1)(k+2)}$

$m_{00} = \frac{4}{2(1)(2)} = 1, \quad m_{10} = \frac{4}{3(1)(2)} = 0.666$

$m_{01} = \frac{2.5+4}{(2)(2)(3)} = 0.54166 \quad m_{20} = \frac{4}{4(1)(2)} = 0.5$

$m_{02} = \frac{5+4}{2(3)(4)} = 0.375 \quad m_{11} = \frac{6.5}{3(2)(3)} = 0.3611.$

→ statistically independent random variables have moments $m_{10} = 2$, $m_{20} = 14$, $m_{02} = 12$, $m_{11} = -6$ Find the moment μ_{22}.

sol:- Given X & Y are statistically independent R.V and also given $m_{10} = 2$, $m_{02} = 12$, $m_{20} = 14$, $m_{11} = -6$

$\mu_{22} = E[(X-\bar{X})^2 + (Y-\bar{Y})^2] = E[(X-\bar{X})^2]\{E(Y-\bar{Y})^2\}$

$= E[X^2 - 2X\bar{X} + \bar{X}^2]\, E[(Y^2) - \bar{Y}^2] = \{E[X^2] - \bar{X}^2\}\{E[Y^2] - \bar{Y}^2\}$

$= \{m_{20} - (m_{10})^2\}\{m_{02} - (m_{01})^2\}$

$= \{14 - (2)^2\}\{12 - (m_{01})^2\}$

$\mu_{22} = 10\{12 - (m_{01})^2\} = 10(12 - 9\} = \underline{30}$

we know $m_{11} = E[XY] = E[X]E(Y)$

$m_{11} = m_{10} \times m_{01}$

$m_{01} = \frac{-6}{2} = -3$

$\boxed{\mu_{22} = 30}.$

→ For two random variable X and Y have the joint density function is

$$f_{XY}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.05\,\delta(x-1)\,\delta(y-3)$$

Find (a) The correlation b) The co-variance c) The correlation coefficient of X and Y c) Are X and Y either uncorrelated or orthogonal

Sol: Given two random variables are discrete R.V hence

$$P(x,y) = f_{XY}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.5\,\delta(x-1)\,\delta(y-3)$$

$P(x,y)$:

y / x	-2	0	1	2	3	P(x)
-1	0	0.15	0	0	0	0.1
0	0	0.1	0	0.1	0	0.2
1	0.4	0	0.2	0	0.05	1.1
P(y)	0.4	0.2	0.2	0.1	0.5	

Fig: Joint probabilities of X and Y.

(a) The correlation $R_{XY} = E[XY] = \sum_i \sum_j P(x_i, y_j)\, x_i y_j$

$$= 0.15(-1)(0) + 0.1(0)(0) + 0.1(0)(2) + 0.4(1)(-2) + 0.2(1)(1) + 0.5(1)(3) = -0.45$$

(b) Co-variance : $\bar{X} = E[X] = \sum_i x_i P(x_i)$

$$\bar{X} = 0.15(-1) + 0.1(0) + 0.1(0) + 0.4(1) + 0.2(1) + 0.05(1) = 0.5$$

$$\bar{Y} = E(Y) = \sum y P(y) = 0.15(0) + 0.1(0) + 0.1(2) + 0.4(2) + 0.2(1) + 0.05(3) = -0.25$$

c) Co-relation coefficient $\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y}$

Now $C_{XY} = \overline{XY} - \bar{X}\bar{Y} = -0.45 - (0.5)(-0.25) = -0.325$

$E\{X^2\} = \overline{X^2} = \sum x^2 P(x,y) = 0.15(-1)^2 + 0.4(1)^2 + 0.2(1)^2 + 0.05(1)^2 = 0.8$

$E\{Y^2\} = \overline{Y^2} = \sum y^2 P(x,y) = 0.1(2^2) + 0.4(-2)^2 + 0.2(1)^2 + 0.05(3^2)$

$= 2.65$

$\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 0.8 - 0.5^2 = 0.55$

$\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 2.65 - (-0.25)^2 = 2.5875$

$$\rho_{XY} = \frac{-0.325}{\sqrt{0.55}\sqrt{2.5875}} = -0.272.$$

and $\rho_{XY} \neq 0$,

d) since $C_{XY} \neq 0$, X and Y are not correlated
X and Y are not orthogonal.

→ For two random variables X and Y.

$f_{XY}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) +$
$0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.05\,\delta(x-1)\,\delta(y-3)$

Find (a) The correlation (b) The variance

c) The correlation coefficient of X and Y

d) Are X and Y either uncorrelated or orthogonal.

→ Show that variance of a weighted sum of uncorrelated random variables equals to the weighted sum of variance of the random variables.

<u>sol</u>: Let $X = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X^n = \sum_{i=1}^{n} \alpha_i X_i$

where α_i represents the weights,

The variance $\sigma_X^2 = E[(X - \bar{X})^2]$

$\bar{X} = E\{X\} = E\left(\sum_{i=1}^{n} \alpha_i X_i\right) = \sum_{i=1}^{n} E[\alpha_i X_i] = \sum_{i=1}^{n} \alpha_i \bar{X}_i$

similarly $X - \bar{X} = \sum_{i=1}^{n} \alpha_i (X_i - \bar{X}_i)$

$\sigma_X^2 = E[(X - \bar{X})^2]$

$= E\left[\sum_{i=1}^{n} \alpha_i (X_i - \bar{X}_i) \sum_{j=1}^{n} \alpha_j (X_j - \bar{X}_j)\right].$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j E\{(X_i - \bar{X}_i)(X_j - \bar{X}_j)\}.$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j C_{X_i X_j} = C_{X_i X_j} = \begin{cases} 0 & \text{for } i \neq j \\ \sigma_X^2 & \text{for } i = j \end{cases}$

$$\sigma_X^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_{X_i}^2$$ (57)

∴ The variance of weighted sum of uncorrelated random variables equals to the weighted sum of variance of the random variables

Joint characteristic function:-

let X & Y are two random variables with joint density function $f_{XY}(x,y)$ then the joint characteristic function is given by

$$\phi_{X,Y}(\omega_1,\omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right] = E\left[e^{j\omega_1 X} e^{j\omega_2 Y}\right].$$

where ω_1 and ω_2 are real numbers. Then

$$\phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, e^{j\omega_1 x + j\omega_2 y}\, dx\, dy \quad -(1)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x} \cdot e^{j\omega_2 y} f_{XY}(x,y)\, dx\, dy \quad -(2)$$

This equation represents the two dimensional Fourier transform of $f_{XY}(x,y)$ with signs of ω_1 and ω_2 are reversed. similarly inverse Fourier transform of $f_{XY}(x,y)$ is given by.

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{X,Y}(\omega_1,\omega_2)\, e^{-j\omega_1 x - j\omega_2 y}\, d\omega_1\, d\omega_2 \quad -(3).$$

→ putting $\omega_2 = 0$, we get the characteristic function random variable 'X' ie $\phi_X(\omega_1) = \phi_{XY}(\omega_1, 0) = E\{e^{j\omega_1 X}\}$

→ putting $\omega_1 = 0$ we get the characteristic function of random variable Y ie $\phi_{XY}(0,\omega_2) = \phi_Y(\omega_2) = E\{e^{j\omega_2 Y}\}$

These are called marginal characteristic functions.

Joint moments:

By using characteristic function we can also find the joint moments and is given by

$$m_{nk} = (-j)^{n+k} \left.\frac{\partial^{n+k} \phi_{XY}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k}\right|_{\omega_1 = 0,\ \omega_2 = 0}$$

for N-random variables ie $X_1, X_2 \ldots X_N$ with characteristic function $\phi_{X_1,X_2\ldots X_N}(\omega_1,\omega_2,\ldots\omega_N)$ then the joint

Moments of N- random variables can be defined as

$$m_{m_1 m_2 \cdots m_N} = (-j)^R \frac{\partial^R \phi_{x_1, x_2, x_3 \cdots x_N}(\omega_1, \omega_2 \cdots \omega_N)}{\partial \omega_1^{m_1} \partial \omega_2^{m_2} \cdots \partial \omega_N^{m_M}} \Bigg|_{\omega_i = 0}$$

here $i = 1, 2 \cdots N$ and $R = m_1 + m_2 + \cdots$

⇒ The random variables X & Y have the joint characteristic function $\phi_{XY}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$ show that X & Y are both zero mean random variables and they are uncorrelated.

sol:- we know that

$$m_{nk} = (-j)^{n+k} \frac{\partial^{n+k} \phi_{XY}(\omega_1 \omega_2)}{\partial \omega_1^n \partial \omega_2^k} \Bigg|_{\text{at } \omega_1 = 0, \omega_2 = 0}$$

$$\bar{X} = E[X] = m_{10} = (-j)^1 \frac{\partial \phi_{XY}(\omega_1 \omega_2)}{\partial \omega_1} \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= (-j) \frac{\partial}{\partial \omega_1} \left\{ e^{-2\omega_1^2 - 8\omega_2^2} \right\} \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= (-j) e^{-2\omega_1^2 - 8\omega_2^2} (-4\omega_1 - 0) \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= j \left(4\omega_1 e^{-2\omega_1^2 - 8\omega_2^2}\right) \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= 0$$

∴ Both X & Y have the zero mean values.

→ If X & Y are uncorrelated then $R_{XY} = E[X] E(Y)$

$$R_{XY} = E[X] E[Y] = m_{11} = (-j)^2 \frac{\partial^2 \phi_{XY}(\omega_1 \omega_2)}{\partial \omega_1 \, \partial \omega_2} \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= -1 \cdot \frac{\partial^2}{\partial \omega_1 \partial \omega_2} \left\{ \left(e^{-2\omega_1^2 - 8\omega_2^2} \right) \right\} \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= (-1) \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \omega_2} \left[e^{-2\omega_1^2 - 8\omega_2^2} \right] \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= -\frac{\partial}{\partial \omega_1} e^{-2\omega_1^2 - 8\omega_2^2} + (-16\omega_2) \Bigg|_{\omega_1 = 0, \omega_2 = 0}$$

$$= 16\omega_2 \frac{\partial}{\partial \omega_1} \left(e^{-2\omega_1^2 - 8\omega_2^2} \right) \Bigg|_{\omega_1 = \omega_2 = 0}$$

$= 16\omega_2 e^{-2\omega_1^2 - 8\omega_2^2}(-4\omega_1)\Big|_{\omega_1=\omega_2=0}$

$= 0$

$\therefore R_{XY} = 0$ & $E(X)E(Y) = 0$ Hence $R_{XY} = E(X)E(Y)$

They are uncorrelated.

→ If X & Y are two independent random variables such that $E[X] = d_1$, variance of $X = \sigma_1^2$, $E[Y] = d_2$, variance of $Y = \sigma_2^2$. Prove that variance of $(X,Y) = \sigma_1^2\sigma_2^2 + d_1^2\sigma_2^2 + d_2^2\sigma_1^2$.

sol:- Given that $E(X) = d_1 = \bar{X}$, variance of $X = \sigma_1^2$

$E[Y] = d_2 = \bar{Y}$, variance of $Y = \sigma_2^2$

consider variance of $(XY) = E[(XY)^2] - \overline{XY}^2$

$= E[X^2Y^2] - (\bar{X}\cdot\bar{Y})^2$

$= E[X^2]E[Y^2] - \bar{X}^2\bar{Y}^2$.

[$\because$ X & Y are independent.

Given $var(X) = \sigma_1^2$

$E[X^2] - \bar{X}^2 = \sigma_1^2$

$E[X^2] = \sigma_1^2 + \bar{X}^2$

$= \sigma_1^2 + d_1^2$

$var(Y) = \sigma_2^2$

$E[Y^2] - \bar{Y}^2 = \sigma_2^2$

$E[Y^2] = \sigma_2^2 + \bar{Y}^2$

$E[Y^2] = \sigma_2^2 + d_2^2$

→ $var(XY) = E[X^2]E[Y^2] - \bar{X}^2\bar{Y}^2$

$= (\sigma_1^2 + d_1^2)(\sigma_2^2 + d_2^2) - (d_1)^2(d_2)^2$

$var(XY) = \sigma_1^2\sigma_2^2 + \sigma_1^2 d_2^2 + d_1^2\sigma_2^2 + \cancel{d_1^2 d_2^2} - \cancel{d_1^2 d_2^2}$

$var(XY) = \sigma_1^2\sigma_2^2 + d_1^2\sigma_2^2 + d_2^2\sigma_1^2$ Hence it is proved.

→ Show that the joint characteristic function of N-independent random variables X_i having characteristic function $\phi_{X_1}(\omega_i)$ is $\phi_{X_1,X_2\cdots X_N}(\omega_1,\omega_2,\cdots\omega_N) = \prod_{i=1}^{N}\phi_{X_i}(\omega_i)$

sol: The characteristic function of N-r.v is given by

$\phi_{X_1,X_2\cdots X_N}(\omega_1\omega_2\cdots\omega_N) = E\left[e^{j\omega_1 X_1 + j\omega_2 X_2 \cdots j\omega_N X_N}\right]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{j\omega_1 x_1 + j\omega_2 x_2 + \cdots + j\omega_N x_N} f_{X_1,X_2\cdots X_N}(x_1,x_2\cdots x_N)\, dx_1\, dx_2\cdots dx_N$

For statistically independent random variables

$$f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdots\cdots f_{x_N}(x_N).$$

$$\Rightarrow \phi_{x_1 x_2 \cdots x_N}(\omega_1 \omega_2 \cdots \omega_N) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{J\omega_1 x_1 + J\omega_2 x_2 + \cdots + J\omega_N x_N} f_{x_1}(x_1) \cdot f_{x_2}(x_2) \cdots f_{x_N}(x_N) \cdot dx_1\, dx_2 \cdots dx_N$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} e^{J\omega_i x_i} f_{x_i}(x_i)\, dx_i$$

$$= \prod_{i=1}^{N} \int_{-\infty}^{\infty} e^{J\omega_i x_i} f_{x_i}(x_i)\, dx_i$$

$$= \prod_{i=1}^{N} E\{e^{J\omega_i x_i}\} = \prod_{i=1}^{N} \phi_{x_i}(\omega_i).$$

$\rightarrow$ For N-random variables show that $|\phi_{x_1 \cdots x_N}(\omega_1 \cdots \omega_N)| \leq \phi_{x_1 \cdots x_N}(0 \cdots 0) = 1$

Sol:- The characteristic function of N-random variables is given by

$$|\phi_{x_1, x_2 \cdots x_N}(\omega_1 \omega_2 \cdots \omega_N)| = |E[e^{J\omega_1 x_1 + J\omega_2 x_2 + \cdots + J\omega_N x_N}]|.$$

$$= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{J\omega_1 x_1 + J\omega_2 x_2 + \cdots + J\omega_N x_N} f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N)\, dx_1 dx_2 \cdots dx_N)\right|$$

$$\leq \left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\{e^{J\sum_{i=1}^{N}\omega_i x_i}\}\right| \left| f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N)\, dx_1 dx_2 \cdots dx_N \right|.$$

$$\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |e^{J\sum_{i=1}^{N}(\omega_i x_i)}|\, |f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N)|\, dx_1 dx_2 \cdots dx_N$$

$$\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} 1 \cdot |f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N)|\, dx_1 dx_2 \cdots dx_N$$

$$\leq \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N)\, e^{J(0)}\, dx_1 dx_2 \cdots dx_N$$

$$\leq \phi_{x_1 x_2 \cdots x_N}(0,0,0) = 1$$

Jointly Gaussian Random variables

Two random variables X & Y are said to be jointly gaussian if their jointly density function is of the form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right]\right]$$

which is sometimes called the bivariate gaussian density. Here

$\bar{x} = E[X]$, $\bar{y} = E[Y]$, $\sigma_x^2 = E[(X-\bar{x})^2]$

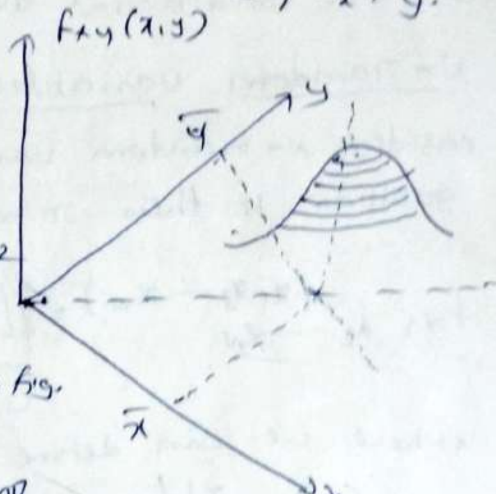
$\sigma_y^2 = E[(Y-\bar{y})^2]$, $\rho = E[(X-\bar{x})(Y-\bar{y})]/\sigma_x\sigma_y$.

→ Jointly gaussian density has the maximum values at $\bar{x}, \bar{y}$.

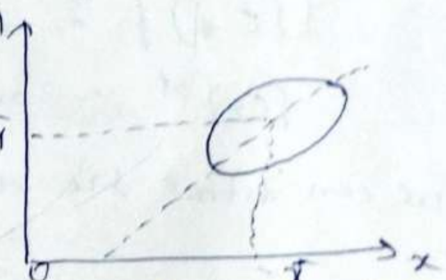
→ The maximum value is

$$f_{XY}(x,y) \le f_{XY}(\bar{x},\bar{y}) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

→ The locus of constant values of $f_{XY}(x,y)$ is an ellipse as shown in fig.



Jointly gaussian density



Locus of density ie ellipse.

→ If X & Y are uncorrelated then correlation coefficient ρ becomes zero then the joint gaussian density function can be written as

$$f_{XY}(x,y) = f_X(x)\cdot f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal densities of X and Y and are given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}}$$

∴ Any uncorrelated gaussian random variables are also statistically independent

Properties :-

- Gaussian random variables are completely defined by their means, variances, and co-variances.
- If Gaussian random variables are uncorrelated then they are also statistically independent.
- Random variables produced by a linear transformations of a gaussian random variables $x_1 x_2 \cdots x_N$ are also gaussian.
- Marginal density functions obtained from a N-variate gaussian density function are also gaussian.
- The conditional density functions are also gaussian.

N-random variables Gaussian density functions :-

consider N-random variables $x_1 x_2 \cdots x_N$ are called jointly gaussian if their joint density function is given by

$$f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N) = \frac{\left|[C_x]^{-1}\right|^{1/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{[x-\bar{x}]^t [C_x]^{-1} [x-\bar{x}]}{2}\right\}$$

where we can define matrices

$[x-\bar{x}]^t$ is transpose of $[x-\bar{x}]$

$|[C_x]|$ = determinant of $[C_x]$

$[C_x]^{-1}$ = inverse of $[C_x]$

we can define the matrices $[x-\bar{x}] = \begin{bmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \\ \vdots \\ x_N-\bar{x}_N \end{bmatrix}$ and

C_x is the covariance matrix

$$[C_x] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

Here the elements of co-variance matrix matrix are given by

$c_{ij} = E\left[(x_i - \bar{x}_i)(x_j - \bar{x}_j) \right]$

$c_{ij} = E\left[(x_i - \bar{x}_i)(x_j - \bar{x}_j) \right] = \begin{cases} \sigma_{x_i}^2 & i = j \\ c_{x_i x_j} & i \neq j \end{cases}$

<u>NOTE</u>:- By putting $N = 2$ in eq (1) we get the jointly gaussian density function of two random variables

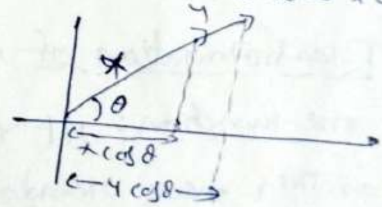
$[x - \bar{x}] = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}$, $C_x = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & \overbrace{\rho \sigma_{x_1} \sigma_{x_2}}^{c_{12}} \\ \underbrace{\rho \sigma_{x_2 x_1}}_{c_{21}} & \sigma_{x_2}^2 \end{bmatrix}$

The correlation coefficient $\rho = \dfrac{c_{12}}{\sigma_{x_1} \sigma_{x_2}} = \dfrac{c_{21}}{\sigma_{x_1} \sigma_{x_2}}$

→ Consider two random variables X_1 and Y_1 related to the R.Vs X and Y by the coordinate rotation

$X_1 = X \cos\theta + Y \sin\theta \qquad Y_1 = Y \cos\theta - X \sin\theta$ where θ is the coordinate rotation angle as shown in fig. If X_1 and Y_1 are Gaussian R.Vs, independent and un-correlated then show that the angle of coordinate rotation is

$\theta = \frac{1}{2} \operatorname{Tan}^{-1}\left[\dfrac{2\rho \sigma_x \sigma_y}{\sigma_x^2 - \sigma_y^2} \right]$



<u>sol</u>: given $X_1 = X\cos\theta + Y \sin\theta$

$Y_1 = Y\cos\theta - X \sin\theta$

if X_1 and Y_1 are uncorrelated then $C_{X_1 Y_1} = 0$

$\therefore \; C_{X_1 Y_1} = E\left[(X_1 - \bar{X}_1)(Y_1 - \bar{Y}_1) \right]$

$= E\left[(X\cos\theta + Y\sin\theta) - (\overline{X\cos\theta + Y\sin\theta}) \right] \left[(Y\cos\theta - X\sin\theta) - (\overline{Y\cos\theta - X\sin\theta}) \right]$

$= E\left\{ (X\cos\theta + Y\sin\theta - \bar{X}\cos\theta - \bar{Y}\sin\theta)(Y\cos\theta - X\sin\theta - \bar{Y}\cos\theta + \bar{X}\sin\theta) \right\}$

$= E\left[\left[(X - \bar{X})\cos\theta + (Y - \bar{Y})\sin\theta \right] \left\{ (Y - \bar{Y})\cos\theta - (X - \bar{X})\sin\theta \right\} \right]$

$= E\left[\left[(X - \bar{X})(Y - \bar{Y})\cos^2\theta - (Y - \bar{Y})(X - \bar{X})\sin^2\theta + (Y - \bar{Y})^2 \sin\theta\cos\theta - (X - \bar{X})^2 \sin\theta\cos\theta \right] \right]$

$= E\left[(X - \bar{X})(Y - \bar{Y})(\cos^2\theta - \sin^2\theta) + E\left[Y - \bar{Y} \right]^2 \frac{1}{2}\sin 2\theta - E\left[(X - \bar{X})^2 \frac{1}{2}\sin 2\theta \right. \right.$

$= E\{(x-\bar{x})(y-\bar{y})\cos 2\theta + \frac{1}{2}(E\{y-\bar{y}\}^2 - E\{(x-\bar{x})^2\}\sin 2\theta$

w.k.t $C_{xy} = E\{(x-\bar{x})(y-\bar{y})\}$

and $\rho = \frac{C_{xy}}{\sigma_x \sigma_y}$ & $C_{xy} = \rho \sigma_x \sigma_y$

$\rightarrow E\{(x-\bar{x})^2\} = \sigma_x^2$ and $E\{(y-\bar{y})^2\} = \sigma_y^2$

$\therefore C_{x_1y_1} = C_{xy}\cos 2\theta + \frac{1}{2}\sin 2\theta(\sigma_y^2 - \sigma_x^2)$

Now $C_{x_1y_1} = 0$

$\therefore \rho\sigma_x\sigma_y \cos 2\theta + \frac{1}{2}\sin 2\theta(\sigma_x^2 - \sigma_y^2) = 0$ (5)

$\sin 2\theta(\sigma_x^2 - \sigma_y^2) = 2\rho\sigma_x\sigma_y\cos 2\theta$

$$\frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta = \frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\rho\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}\right) \quad \text{Proved}$$

Transformation of multiple random variables:-

one function:- let $x_1, x_2 \cdots x_N$ are "N" random variables and another new random variable "y" is given by $y = g(x_1, x_2 \cdots x_N)$ — (1)

The distribution function y is given by

$$F_Y(y) = P\{Y \le y\} = P\{g(x_1, x_2 \cdots x_N) \le y\}$$

$$= \int \cdots \iint_{\{g(x_1, x_2 \cdots x_N) \le y\}} f_{x_1, x_2 \cdots x_N}(x_1, x_2 \cdots x_N)\, dx_1\, dx_2 \cdots dx_N \quad \longrightarrow (2)$$

The density function of y is given by

$$f_Y(y) = \frac{d}{dy}\int \cdots \iint_{\{g(x_1, x_2 \cdots x_N) \le y\}} f_{x_1, x_2 \cdots x_N}(x_1, x_2 \cdots x_N)\, dx_1 \cdots dx_N$$

Multiple functions:- consider N-random variables $x_1, x_2 \cdots x_N$. Now define another set of random variables $y_1, y_2 \cdots y_N$.

Now $Y_n = T_n(x_1, x_2, x_3 \cdots x_N),\ n = 1, 2, 3 \cdots N$ — (1)

Now $x_M = T_M^{-1}(y_1, y_2 \dots y_N)$ $M = 1, 2, \dots N$ — ②

If R_x and R_y are closed regions of x & y respectively then the joint density functions are given by

$$\int_{R_x} \cdots \int f_{x_1 \dots x_N}(x_1, x_2 \dots x_N)\, dx_1 \dots dx_N.$$

$$= \int \cdots \int_{R_y} f_{y_1, y_2 \dots y_N}(y_1, y_2 \dots y_N)\, dy_1\, dy_2 \dots dy_N \quad \text{—③}$$

Apply the transformations

$$\int_{R_x} \cdots \int f_{x_1 \dots x_N}(x_1 \dots x_N)\, dx_1 \dots dx_N = \int_{R_y} \cdots \int f_{x_1 \dots x_N}(x_1 = T_1^{-1}(y_1) \dots \dots x_N = T_N^{-1}(y_N))\, |J|\, dy_1 \dots dy_N. \quad \text{—④}$$

where $|J|$ is the magnitude of Jacobian function

$$J = \begin{bmatrix} \frac{\partial T_1^{-1}}{\partial y_1} & \frac{\partial T_1^{-1}}{\partial y_2} & \cdots & \frac{\partial T_1^{-1}}{\partial y_N} \\ \vdots & & & \\ \frac{\partial T_N^{-1}}{\partial y_1} & \frac{\partial T_N^{-1}}{\partial y_2} & \cdots & \frac{\partial T_N^{-1}}{\partial y_N} \end{bmatrix}$$

substitute eq ④ in eq ③

$$\int_{R_y} \cdots \int f_{y_1 \dots y_N}(y_1 \dots y_N)\, dy_1 \dots dy_N = \int_{R_y} f_{x_1 \dots x_N}(x_1 = T_1^{-1} \dots x_N = T_N^{-1})\, |J|\, dy_1 \dots dy_N.$$

<u>Linear</u> <u>Transformation</u> <u>of a</u> <u>gaussian</u> random <u>variables</u>:

" " " " Randoms produces another gaussian random variables

$$[C_y] = [T][C_x][T]^t \quad \text{—①}$$

<u>Problem</u>:- Two random variables x_1 and x_2 have zero means and variances $\sigma_{x_1}^2 = 4$ and $\sigma_{x_2}^2 = 9$. Their covariance is $C_{x_1 x_2} = 3$. If x_1 and x_2 linearly transformed to new variables y_1 and y_2 according to $y_1 = x_1 - 2x_2$, $y_2 = 3x_1 + 4x_2$ — find the mean and variances and co-variances of y_1 & y_2.

sol: Given x_1 & x_2 are zero means and gaussian
Hence y_1 & y_2 also zero mean and gaussian
Given $y_1 = x_1 - 2x_2$, $y_2 = 3x_1 + 4x_2$.

From this $T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $[T]^t = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$

we know that $[C_x] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{x_1}^2 & c_{x_1x_2} \\ c_{x_2x_1} & \sigma_{x}^2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$

$$[C_y] = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 3-18 \\ 12+12 & 9+36 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 24 & 45 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2+30 & -6-60 \\ 24-90 & 72+180 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}$$