UNIT-2

Operations on one Random variable - Expectations :-

Introduction:- The process of averaging when a random variable is involved is called expectation. It is denoted by $E(x)$, or $\bar{X}$. And it is read as Expected value of "X" or the mean value of "X" are also called as statistical average of "x".

Expected value of a random variable:-

If "x" be a continuous random variable with prob density function $f_X(x)$ then the expected value of X (or) mean value of X is defined as

$$\bar{X} = E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \quad -① $$

If "x" be a discrete random variable with the set of elements $\{x_1, x_2 ---- x_N\}$ and a set of corresponding probabilities $\{P(x_1), P(x_2) --- P(x_N)\}$ respectively. Then the expected value of X is

$$\sum_{i=1}^{N} x_i P(x_i) \quad — ②$$

NOTE:- Here $P(x_i)$ denotes the prob mass function.

case (i):- Let us consider all are equiprobable elements ie

$$P(x_1) = P(x_2) = P(x_3) = ----- = P(x_N) = \frac{1}{N}.$$

Then $E(X) = \sum_{i=1}^{N} x_i \cdot \frac{1}{N} = \frac{1}{N}(x_1 + x_2 + ---- + x_N)$. which is arithmatic mean value of "x".

→ Find the expected (mean) value of a exponential distributed R.V.

sol:- we know that the prob density function of exponential is given by

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & \text{for } x \geq a \\ 0 & x < a \end{cases}$$

The expected value of a R.V is $E(x) = \bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{b} e^{-(x-a)/b} dx =$$

$$= \frac{1}{b}\left[x \cdot \frac{e^{-(x-a)/b}}{-\frac{1}{b}} - \int \frac{e^{-(x-a)/b}}{-\frac{1}{b}} \right]_a^{\infty}$$

$$= \left\{ -x e^{-(x-a)/b} + \frac{e^{-(x-a)/b}}{+1/b} \right\} \Big|_a^{\infty}$$

$= a+b$

The expected or mean value of Exponential R.V is $a+b$.

⇒ Find the mean value of uniform distributed random variable.

sdu:- The prob density function of a r.v is $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{a+b}{2}.$$

Expected value of a function of a random variable:-

If $g(x)$ is a real function of "x" then the expected value of $g(x)$ for a continuous random variable 'x' is defined as

$$E(g(x)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \quad - (3)$$

If X is a discrete random variable then

$$E[g(x)] = \sum_{i=1}^{N} g(x_i)\, P(x_i) \quad - (4)$$ where "N" may be infinite.

Conditional expected value:-

Let "X" is a continuous random variable with conditional pdf $f_X(x|B)$, where B is any event defined in the sample space then the conditional expected value is given by

$$E\{X|B\} = \int_{-\infty}^{\infty} x \cdot f_X(x|B)\, dx \quad - (1)$$

If the event B depends on X such that $B = \{X \le b\}$ for $-\infty < b < \infty$ — (2)

we know that

$$f_X(x/X \le b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} & \text{for } x < b \\ 0 & \text{else} \end{cases} \quad - (3)$$

substitute (3) in (1)

$$E[X/X \le b] = \frac{\int_{-\infty}^{b} x f_X(x)\, dx}{\int_{-\infty}^{b} f_X(x) \cdot dx} \quad - (4)$$

Properties of Expectation :-

* If a random variable "x" is constant ie $x = a$ then $E[a] = a$

Proof:- $E[x] = \bar{x} = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx$

$$E[a] = \int_{-\infty}^{\infty} a f_X(x)\, dx = a \int_{-\infty}^{\infty} f_X(x)\, dx = a \times 1 = a.$$

* If a is constant then $E[ax] = a E[x]$ "a" is constant.

Proof:- $E[ax] = \int_{-\infty}^{\infty} a x f_X(x)\, dx = a \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = a E[x].$

* If a & b real constants then $E[ax+b] = a E[x] + b.$

Proof:- $E[ax+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x) \cdot dx =$

$$= a \int_{-\infty}^{\infty} x f_X(x)\, dx + b \int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= a E[x] + b$$

* If $g_1(x)$ and $g_2(x)$ are two functions of a random variable x then $E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)].$

Proof:- $E[g_1(x) + g_2(x)] = \int_{-\infty}^{\infty} g_1(x) f_X(x) \cdot dx + \int_{-\infty}^{\infty} g_2(x) f_X(x)\, dx$

$$= E[g_1(x)] + E[g_2(x)].$$

* $|E[g(x)]| \leq E[|g(x)|].$

Problems:- The density function of random variable x is

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else} \end{cases}$$ then find a) $E[x]$ b) $E[x^2]$ c) $E[(x-1)^2]$

Sol:- Given $f_X(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else where} \end{cases}$

$$E[x] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{\infty} x e^{-x}\, dx = \int_{0}^{\infty} x e^{-x} dx$$

$$= -x e^{-x} - e^{-x} = -e^{-x}(x+1)\Big|_0^{\infty} = 1$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_{0}^{\infty} x^2 e^{-x} dx$$

$= -x^2 e^{-x} + \int e^{-x} 2x \, dx$

$= -x^2 e^{-x} + 2 \int e^{-x} \cdot x \, dx$

$= -x^2 e^{-x} + 2 \{ -x e^{-x} - e^{-x} \} \Big|_0^\infty$

$= -x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} \Big|_0^\infty = -e^{-x} \{ x^2 + 2x + 2 \} \Big|_0^\infty$

$E[x^2] = 2$

c) $E[(x-1)^2] = \int_{-\infty}^{\infty} (x-1)^2 f_X(x) \cdot dx = \int_0^\infty (x-1)^2 e^{-x} dx$

$= \int_0^\infty (x^2 - 2x + 1) e^{-x} dx = \int_0^\infty x^2 e^{-x} - 2 \int_0^\infty x e^{-x} + \int_0^\infty e^{-x}$

$= 2 - 2 + 1 = \underline{1}$

→ If X be a discrete R.V with $p(x)$ is given by

x =	-2	-1	0	1	2
$P(x)$ =	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$

Find (i) $E(X)$ (ii) $E[2X+3]$ (iii) $E[X^2]$ (iv) $E[(2X+1)^2]$.

(i) $E(X) = \sum_{i=1}^{N} P(x_i) x_i$ if X is a discrete Random variable

$= \sum_{i=-2}^{2} x_i P(x_i) = -2 \left(\frac{1}{5}\right) - 1 \left(\frac{2}{5}\right) + 0 + \frac{1}{5} + \frac{2}{5} = \underline{-0.3}$

(ii) $E[2X+3] = 2E[X] + 3 = 2(-0.3) + 3 = +2.4$

(iii) $E[X^2] = \sum_{i=-2}^{2} x_i^2 P(x_i)$

$= 4 \left[\frac{1}{5}\right] + \frac{2}{5} + 0 + \frac{1}{10} + \frac{4}{5} = \frac{21}{10}$

(iv) $E[(2X+1)^2] = \sum_{i=-2}^{2} (2X+1)^2 P(x_i)$

$E[4X^2 + 4X + 1] = \sum_{i=-2}^{2} 4[E(X^2)] + 4E(X) + 1$

$= 4\left(\frac{21}{10}\right) + 4\left(\frac{-3}{10}\right) + 1$

$= \frac{82}{10}$.

Moments:

There are two types of moments for a function of a random variable x

- → Moments about the origin
- → Moments about the mean value (or) central moments.

Moments about the origin:

The function $g(x) = x^n$, $n = 0, 1, 2 \dots$ ①.

Then the Expected value of function $g(x)$ is called the moments about the origin of a random variable "x". It is denoted by m_n where "n" denotes the order of the moments.

Mathematically the nth moment is defined as

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx \quad - ②$$

→ For a discrete random variable $E[x^n] = \sum_{i=1} x_i^n P(x_i)$ - ③.

→ If $n=0$, $m_0 = \int_{-\infty}^{\infty} x^0 f_x(x)\, dx = \int_{-\infty}^{\infty} f_x(x)\, dx = 1$

∴ The zeroth moment of "x" equals to total area under the pdf curve i.e $m_0 = 1$

→ If $n=1$, $m_1 = \int_{-\infty}^{\infty} x . f_x(x)\, dx = E[x] = \bar{x}$

The first moment is a mean value of x

→ If $n=2$, $m_2 = \int_{-\infty}^{\infty} x^2 f_x(x) . dx = E[x^2] = \bar{x}^2$

The second moment of "x" equal to mean square value.

NOTE: The mean value of a random variable can be -ve, but mean square is always non-negative i.e $E[x^2] \geq 0$.

Central Moments:

Moments about the mean value of "x" are called central moments.

The function $g(x) = (x - \bar{x})^n$ where $n = 0, 1, 2 \dots$ ①.

where $\bar{x}$ is the mean of the random variable x. then the expected value of a function $g(x)$ is called Moments about the mean value (or) also called as central moments. It is denoted by μ_n.

where "m" indicates the order of the moment.

mathematically the mth central moment is

$$\mu_m = E[(X-\bar{X})^m] = \int_{-\infty}^{\infty} (x-\bar{x})^m f_X(x)\,dx \quad -②$$

For discrete R.V $\mu_m = \sum_i (x_i - \bar{x})^m P(x_i) \quad -③$

The zeroth central moment of X is "1" ie $\mu_0 = m_0 = 1$, Area under the pdf curve.

→ If $m=1$, $\mu_1 = E[X-\bar{X}] = \int_{-\infty}^{\infty} (x-\bar{x}) f_X(x)\,dx$,

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx - \bar{x}\int_{-\infty}^{\infty} f_X(x)\,dx \quad \therefore \bar{x} - \bar{x} = 0.$$

The first central moment of 'x' equals to zero.

Variance:- Variance of the density function $f_X(x)$ for a random variable "x" is defined as the second order central moment of a random variable "x". and it is denoted by σ_x^2. and is given by

$$\sigma_x^2 = \mu_2 = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\,dx \quad -①$$

For discrete R.V $\sigma_x^2 = \sum_i^N (x_i - \bar{x})^2 P(x_i) \quad -②$.

The +ve square root of varience ie σ_x is called standard deviation

ie $\sigma_X = \sqrt{E[(X-\bar{X})^2]} = E[(X-\bar{X})^2]^{1/2}$

→ varience can also find from the knowledge of first and second moments.

→ consider $\sigma_x^2 = var(X) = E[(X-\bar{X})^2]$

$$= E[X^2 - 2X\bar{X} + \bar{X}^2]$$

$$= E[X^2] + \bar{X}^2 E[1] - 2\bar{X}E[X]$$

$$= E[X^2] + \bar{X}^2 - 2\bar{X}\bar{X}$$

$$= E[X^2] + \bar{X}^2 - 2\bar{X}^2$$

$$= E[X^2] - \bar{X}^2 = m_2 - m_1^2$$

m_1 = 1st order moment $E[X]$

m_2 = second order moment

$$\boxed{\sigma_X^2 = var(X) = m_2 - m_1^2}$$

Skew:- The skew of the density function $f_X(x)$ for a R.V "X" is defined as the third central moment of R.V "X" and is given by

$$\mu_3 = E[(X-\bar{X})^3] = \int_{-\infty}^{\infty} (x-\bar{X})^3 f_X(x)\,dx \quad \text{—(1)}.$$

The skew of a density function is a measure of the probability density function $f_X(x)$ about mean ie $x = \bar{X} = m$

If a density is symmentric about $X = \bar{X}$ then the skew is zero.

skewness:- The ratio of 3rd central moment to the cube of standard deviation is called skewness of the density function (γ) coefficient of skewness

$$\text{skewness} = \frac{\mu_3}{\sigma_X^3} = \frac{E[(X-\bar{X})^3]}{E[(X-\bar{X})^{3/2}]}.$$

⇒ Third order central moment in terms of Mean value and variance of
Consider 3rd order central moment $\mu_3 = E[(X-\bar{X})^3]$

$$\mu_3 = E[X^3 - \bar{X}^3 - 3X^2\bar{X} + 3X\bar{X}^2]$$

$$= E[X^3] - \bar{X}^3 E[1] - 3\bar{X}E[X^2] + 3\bar{X}^2 E[X]$$

$$= E[X^3] - \bar{X}^3 - 3\bar{X}\,\overline{X^2} + 3\bar{X}^2\bar{X}$$

$$= E[X^3] - \bar{X}^3 - 3\bar{X}\,\overline{X^2} + 3\bar{X}^3$$

$$= E[X^3] + 2\bar{X}^3 - 3\bar{X}\,\overline{X^2}$$

$$= E[X^3] + 2\bar{X}^3 - 3\bar{X}[\sigma_X^2 + \bar{X}^2]$$

$$= E[X^3] + 2\bar{X}^3 - 3\bar{X}\sigma_X^2 - 3\bar{X}^3$$

$$\left[\begin{array}{l} m_2 = E[X^2] \\ m_1 = E[X] \\ m_1^2 = (E(X))^2 \end{array}\right.$$

w.k.T

$$\sigma_X^2 = E[X^2] - \bar{X}^2$$

$$\overline{X^2} = \sigma_X^2 + \bar{X}^2$$

$$\boxed{\mu_3 = E[X^3] - 3\bar{X}\sigma_X^2 - \bar{X}^3} = \mu_3 = \boxed{m_3 - 3m_1\sigma_X^2 - m_1^3}$$

Properties of variance:-

→ If a is any constant then $\text{var}(aX) = a^2\,\text{var}(X)$.

Proof:- we know that $\text{var}(X) = E[(X-\bar{X})^2]$

$$\rightarrow \text{var}(aX) = E[(aX - a\bar{X})^2] = E[a^2(X-\bar{X})^2]$$

$$= a^2 E[(X-\bar{X})^2] = a^2\,\text{var}(X).$$

~~var~~ $\text{var}(X) = E[X^2] - \bar{X}^2 = m_2 - m_1^2$

Here $E[X^2]$ = second moment = m_2 = mean square value.

→ If "x" is a R.V then $var(ax+b) = a^2 var(x)$.

Proof:- $var(x) = E[(x-\bar{x})^2]$

$var(ax+b) = E[\{(ax+b) - (\overline{ax+b})\}^2]$.

$= E[\{(ax+b) - a\bar{x}+b)\}^2] = E[\{ax+\cancel{b}-a\bar{x}-\cancel{b}\}^2]$

$= a^2 E[(x-\bar{x})^2] = a^2 var(x)$.

→ If x & y are two independent random variables then

$var(x+y) = var(x) + var(y)$.

$var(x) = E[(x-\bar{x})^2]$

$var(x+y) = E[\{(x+y) - (\overline{x+y})\}^2] = E[(x+y-\bar{x}-\bar{y})^2]$.

$= E[(x-\bar{x})^2 + (y-\bar{y})^2 + 2(x-\bar{x})(y-\bar{y})]$.

$= E[(x-\bar{x})^2] + E[(y-\bar{y})^2] + 2E[(x-\bar{x})(y-\bar{y})]$.

Here x and y are two independent random variables then

$E[(x-\bar{x})(y-\bar{y})] = E[(x-\bar{x})E(y-\bar{y})]$.

$= [E[x] - \bar{x}][E(y) - \bar{y}]$

$= (\bar{x}-\bar{x})(\bar{y}-\bar{y}) = 0$

$\therefore var(x+y) = E[(x-\bar{x})^2 + E[y-\bar{y}]^2 + 0$

$= var(x) + var(y)$.

→ If x and y are two independent random variables then

$E[x-y] = var(x) + var(y)$.

$var(x-y) = E[[(x-y) - (\overline{x-y})]^2]$.

$= E[[(x-y) - (\bar{x}-\bar{y})]^2]$.

$= E[[(x-\bar{x}) - (y-\bar{y})]^2]$.

$= E[(x-\bar{x})^2 + E(y-\bar{y})^2 - 2E[(x-\bar{x})(y-\bar{y})]$.

$= E[(x-\bar{x})^2 + E(y-\bar{y})^2 + 0$

$\therefore$ $var(x-y) = var(x) + var(y)$

$\because$ $var(x-y) = var(x+(-y))$

$= var(x) + var(-y) = var(x) + (-1)^2 var(y).$

$var(ax) = a^2 var(x)$

$$\boxed{var(x-y) = var(x) + var(y)}.$$

→ Find the expected value of the number on a die when thrown.

sol: Let "x" be a R.V which takes the values when die is thrown.

$x = \{1 \ 2 \ 3 \ 4 \ 5 \ 6\}$

$x = x_i$	1	2	3	4	5	6
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E[x] = \sum_{i=1}^{6} x_i P(x_i)$

$= 1(\frac{1}{6}) + 2(\frac{1}{6}) + 3(\frac{1}{6}) + 4(\frac{1}{6}) + 5(\frac{1}{6}) + 6(\frac{1}{6}) = 7/2.$

→ In an experiment when two dies are thrown. find the expected value of sum of the numbers shown on the die.

sol:- Let "x" be a R.V which denotes the sum of the numbers shown on the die when two dies are throughtum then the possible outcomes are 36.

S = { (1,1) (1,2) (1,3) (1,4), (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6) }

Let x be a R.V which denotes the sum of the numbers shown on die.

$x = \{2, 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11, 12\}$

$x = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$\therefore E[x] = \sum x_i p(x_i).$

$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} = 7.$

→ The PDF of a R-V X is given by $f_X(x) = \begin{cases} 0.3507\sqrt{x} & 0<x<3 \\ 0 & \text{else} \end{cases}$

Find (i) mean (ii) mean of the square (iii) varience of the R.V.

sol: Given $f_X(x) = \begin{cases} 0.3507\sqrt{x} & \text{for } 0<x<3 \\ 0 & \text{else} \end{cases}$

(i) mean value

$$\text{Mean} = E\{X\} = \bar{X} = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx$$

$$= \int_0^3 x \cdot 0.3507\sqrt{x}\, dx = 0.3507 \int_0^3 x^{3/2}\, dx.$$

$$= 0.3507 \cdot \frac{2}{5} \cdot x^{5/2} \Big/_0^3 = 2.18674.$$

(ii) Mean square:

$$\text{Mean square} = \overline{X^2} = E\{X^2\} = \int_{-\infty}^{\infty} x^2 \cdot 0.3507\sqrt{x}\, dx.$$

$$= \int_0^3 x \cdot 0.3507\, x^{5/2}\, dx = 0.3507 \cdot \frac{2}{7} \cdot x^{7/2} \Big/_0^3$$

$$= 4.68589$$

(iii) varience $= \sigma_x^2 = E\{X^2\} - \bar{X}^2$

$$= 4.68589 - 2.18674 = 0.$$

→ Let "x" be a random variable defined by the density function

$$f_X(x) = \begin{cases} \frac{5}{4}(1-x^4) & 0 < x \le 1 \\ 0 & \text{else where of "x".} \end{cases}$$

find (i) $E[x]$ (ii) $E[x^2]$ (iii) $E[4x+2]$ and (iv) varience.

sol: Given $f_X(x) = \begin{cases} \frac{5}{4}(1-x^4) & 0<x\le 1 \\ 0 & \text{else} \end{cases}$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_0^1 x \cdot \frac{5}{4}(1-x^4)\, dx = \frac{5}{4}\int_0^1 x - x^5\, dx$$

$$= \frac{5}{4}\left[\frac{x^2}{2} - \frac{x^6}{6}\right]_0^1 = \frac{5}{4}\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{5}{4}\left[\frac{1}{3}\right] = 0.4166.$$

(ii) $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \frac{5}{4}\int_0^1 x^2(1-x^4)\, dx = \frac{5}{4}\left\{\frac{x^3}{3} - \frac{x^7}{7}\right\}\Big/_0^1$

$$= \frac{5}{4}\left\{\frac{1}{3} - \frac{1}{7}\right\} = \frac{5}{21}.$$

(iii) $E[4x+2] = 4E[x] + 2 = 4\left(\frac{5}{12}\right) + 2 = \frac{11}{3}$

(iv) varience $\sigma_x^2 = E[x^2] - \bar{x}^2 = \frac{5}{21} - \left(\frac{5}{12}\right)^2 = 0.0644 8.$

Exponential functions :-

$\int e^{ax} dx = \frac{e^{ax}}{x}$ "a" real or complex

$\int x e^{ax} dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right]$ "a" real or complex

$\int x^2 e^{ax} dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$ a real or complex.

$\int x^3 e^{ax} dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right]$ a real or complex

$\int e^{ax} \cdot \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$.

Definite integrals

$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx} dx = \frac{\sqrt{\pi}}{a} e^{b^2/4a^2} \quad a>0$$

$$\int_0^{\infty} x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$$

$$\int_0^{\infty} Sa(x) dx = \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

$$\int_0^{\infty} Sa^2(x) dx = \frac{\pi}{2}$$

$$\sum_{m=1}^{N} m = \frac{N(N+1)}{2}, \quad \sum_{N=1}^{N} m^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{m=1}^{N} m^3 = \frac{N^3 [N+1]^2}{4}$$

$$\sum_{m=0}^{N} x^m = \frac{x^{N+1}-1}{x-1}$$

$$\sum_{m=0}^{N} e^{j(\theta+m\phi)} = \frac{\sin[(N+1)\theta/2]}{\sin(\theta/2)} e^{j(\theta+(N\phi/2)}$$

$$\sum_{m=0}^{N} \frac{N!}{m!(N-m)!} x^m y^{N-m} = (x+y)^N$$

$$\sum_{m=0}^{N} \binom{N}{m} = \sum_{m=0}^{N} \frac{N!}{m!(N-m)!} = 2^N$$

$\sum_{n=N_1}^{N_2} \omega^n = \frac{\omega^{N_1} + \omega^{N_2+1}}{1-\omega}$ ($N_2 > N_1$ and ω is real or complex)..

Trigonometric functions:.

$\int uv = u\int v\,dv - \int v\,du$

$\int \cos x\,dx = \sin x$

$\int x\cos x\,dx = \cos x + x\sin x$

$\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x$

$\int \sin x\,dx = -\cos x$

$\int x\sin x\,dx = \sin x - x\cos x$

$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x$

→ Find the expected value of the function $g(x) = x^2$, where X is a R.V defined by the density function $f_X(x) = ae^{-ax}u(x)$ where a is constant.

sol:. $E[g(x)] = E[X^2] = \int_{-\infty}^{\infty} x^2 a e^{-ax} u(x)\,dx = a\int_0^{\infty} x^2 e^{-ax}\,dx$

$= a\left[x^2\frac{e^{-ax}}{-a} + \int \frac{e^{-ax}}{a}2x\right]_0^{\infty}$

$= a\left[-\frac{x^2e^{-ax}}{a} + \frac{2}{a}\left[\frac{xe^{-ax}}{-a} + \int \frac{e^{-ax}}{a}\cdot 1\right\}\right.$

$= -x^2e^{-ax} - \frac{2x}{a}e^{-ax} + \frac{2}{a}\frac{e^{-ax}}{-a}\Big|_0^{\infty} = \frac{2}{a^2}$.

→ A R.V "x" has a pdf $f_X(x) = \begin{cases}\frac{1}{2}\cos x & -\frac{\pi}{2} < x < \frac{\pi}{2}\\ 0 & \text{else}\end{cases}$

find the mean value of the function $g(x) = 4x^2$

sol: Mean value $= E[g(x)] = E[4x^2] = \int_{-\pi/2}^{\pi/2} 4x^2 \cdot \frac{1}{2}\cos x$

$= 2\int_{-\pi/2}^{\pi/2} x^2\cos x\,dx$

$= 2\left[2x\cdot\cos x + (x^2-2)\sin x\right]\Big|_{-\pi/2}^{\pi/2}$

$= 2\left[0 + \left(\frac{\pi^2}{4}-2\right) - \left(\frac{\pi^2}{4}-2\right)(-1)\right]$

$= 4\left\{\frac{\pi^2}{4}-2\right\} = \pi^2 - 8$.

(2)

→ Let X be a R.V defined by the density function $f_X(x) = \begin{cases} \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right) & -4 \le x \le 4 \\ 0 & \text{else} \end{cases}$

find $E[3X]$ and $E[X^2]$.

$$E[3X] = 3E[X] = 3\int_{-4}^{4} x \cdot \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right) dx.$$

$$= \frac{6\pi}{16}\left\{ x \cdot \sin\left(\frac{\pi x}{8}\right)\cdot\left(\frac{8}{\pi}\right) - \int \sin\left(\frac{\pi x}{8}\right)\cdot\frac{8}{\pi}\cdot dx \right\}.$$

$$= \frac{\cancel{3}\pi}{16}\left[x\sin\left(\frac{\pi x}{8}\right)\cdot\left(\frac{8}{\pi}\right) + \frac{8}{\pi}\cdot\frac{8}{\pi}\cdot\cos\left(\frac{\pi x}{8}\right)\right\}_{0}^{4}$$

$$= \frac{3}{2}\left[x\sin\left(\frac{\pi x}{8}\right) + \frac{8}{\pi}\cos\left(\frac{\pi x}{8}\right)\right\}_{0}^{4}$$

$$= \frac{3}{8}\left[4\sin\left(\frac{\pi}{2}\right) + \frac{8}{\pi}\cos\left(\frac{\pi}{2}\right)\right] - \left\{0 + \frac{8}{\pi}\cos(0)\right\}_{-4}^{4}$$

$$= \cancel{\frac{3}{2}\left[4 + \frac{8}{\pi}\right] = 12 + \frac{24}{\pi}}$$

$$= \frac{3}{2}\left[4\cdot\sin\left(\frac{\pi}{2}\right)\right] + \frac{8}{\pi}\cos\left(\frac{\pi}{2}\right) - \left\{4\sin\frac{\pi}{2} + \frac{8}{\pi}\cos\frac{\pi}{2}\right) = 0 //.$$

$$\rightarrow E\{X^2\} = \int_{-4}^{4} x^2\cdot\frac{\pi}{16}\cdot\cos\left(\frac{\pi x}{8}\right) = \frac{2\pi}{16}\int_{0}^{4} x^2\cos\left(\frac{\pi x}{8}\right) dx.$$

$$= \frac{\pi}{8}\left[x^2\sin\frac{\pi x}{8}\cdot\frac{8}{\pi} - \int \frac{8}{\pi}\cdot\sin\frac{\pi x}{8}\cdot 2x\right).$$

$$= \frac{\pi}{8}\left[x^2\sin\left(\frac{\pi x}{8}\right) - 2\int \sin\frac{\pi x}{8}\cdot x\, dx\right.$$

$$= x^2\sin\left(\frac{\pi x}{8}\right) - \frac{16}{\pi}\left[-x\cdot\cos\left(\frac{\pi x}{8}\right) + \int \cos\left(\frac{\pi x}{8}\right) dx\right]$$

$$= x^2\sin\left(\frac{\pi x}{8}\right) + \frac{16}{\pi}x\cdot\cos\left(\frac{\pi}{8}x\right) - \frac{16}{\pi}\cdot\left[\sin\frac{\pi x}{8}\cdot\frac{8}{\pi}\right].$$

$$= x^2\sin\left(\frac{\pi x}{8}\right) + \frac{16x}{\pi}\cos\left(\frac{\pi}{8}x\right) - \frac{144}{\pi^2}\sin\frac{\pi x}{8}\Big|_{0}^{4}$$

$$= 16\sin\left(\frac{\pi}{2}\right) + \frac{64}{\pi}\cos\left(\frac{\pi}{2}\right) - \frac{128}{\pi^2}\sin\left(\frac{\pi}{2}\right)$$

$$= 16 + \frac{\cancel{144}\ 128}{\pi^2}$$

→ A R.V "x" has a density function $f_X(x) = \frac{1}{a} e^{-b|x|}$ for $-\infty \le x \le \infty$. Find $E[X]$ & $E[X^2]$.

sol:- Given $f_X(x) = \frac{1}{a} e^{-b|x|}$ for $-\infty \le x \le \infty$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{a} e^{-b|x|} dx = \frac{1}{a}\int_{-\infty}^{\infty} x e^{-b|x|} dx.$$

$$= \frac{1}{a}\int_{-\infty}^{0} x e^{bx} dx + \frac{1}{a}\int_{0}^{\infty} x e^{-bx} dx.$$

$$= \frac{1}{a}\left[x \cdot \frac{e^{bx}}{b} - \int \frac{e^{bx}}{b}\right] + \frac{1}{a}\left[x\frac{e^{-bx}}{-b} + \int \frac{e^{-bx}}{b}\right].$$

$$= \frac{1}{a}\left[\frac{x}{b}e^{bx} - \frac{e^{bx}}{b^2}\right]\Big|_{-\infty}^{0} + \frac{1}{a}\left[\frac{-x}{b}e^{-bx} - \frac{1}{b^2}e^{-bx}\right]\Big|_{0}^{\infty}$$

$$= \frac{1}{a}\left[-\frac{1}{b^2}\right] + \frac{1}{a}\left[\frac{1}{b^2}\right] = 0$$

(ii) $$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{a} e^{-b|x|}$$

$$= \frac{1}{a}\int_{-\infty}^{0} x^2 e^{bx} dx + \frac{1}{a}\int_{0}^{\infty} x^2 e^{-bx} dx$$

$$= \frac{1}{a}\left[e^{bx}\left(\frac{x^2}{b} - \frac{2x}{b} + \frac{2}{b^3}\right)\right]\Big|_{-\infty}^{0} + \frac{1}{a}\left[e^{-bx}\left[\frac{-x^2}{b} - \frac{2x}{b^2} - \frac{2}{b^3}\right]\right]_0^\infty$$

$$= \frac{1}{a}\left[\frac{2}{b^3}\right] + \frac{1}{a}\left[\frac{2}{b^3}\right] = \frac{4}{ab^3}$$

→ Gaussian density function:-

we know that $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$ for all x

Mean value:- $$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx.$$

Let $\frac{x-a_x}{\sigma_x} = t$, $\Rightarrow x = a_x + \sigma_x t$. $dx = \sigma_x dt$

$x = -\infty$, $\Rightarrow t = -\infty$

$x = \infty$, $t = \infty$

$$\left[\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}\right]$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot e^{-t^2/2} dx \frac{1}{\sqrt{2\pi\sigma_x^2}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (a_x + \sigma_x t) e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (a_x + \sigma_x t) e^{-t^2/2} dt = \frac{1}{\sqrt{2\pi}}\left[a_x\int_{-\infty}^{\infty} e^{-t^2/2} dt + \sigma_x\int_{-\infty}^{\infty} t e^{-t^2/2} dt\right]$$

$$= \frac{1}{\sqrt{2\pi}} a_x \cdot \sqrt{2\pi} + 0 = a_x.$$ //

Varience:- $\sigma_x^2 = E[x^2] = \overline{x^2}$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} \text{ for all } x. = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx.$$

Let $\frac{x-a_x}{\sigma_x} = t$, $\Rightarrow$ $x = a_x + \sigma_x t$

$dx = \sigma_x \cdot dt$

$$E[x^2] = \int_{-\infty}^{\infty} (a_x + \sigma_x t)^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-t^2/2} \sigma_x \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (a_x^2 + \sigma_x^2 t^2 + 2\sigma_x a_x t) e^{-t^2/2} dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_x^2 e^{-t^2/2} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x^2 t^2 e^{-t^2/2} dt + 2\sigma_x a_x \int_{-\infty}^{\infty} t e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot a_x^2 \cdot \sqrt{2\pi} + \frac{1}{\sqrt{2\pi}} \cdot \sigma_x^2 \int_{-\infty}^{\infty} t^2 e^{-t^2/2} dt + 2\sigma_x a_x \int_{-\infty}^{\infty} t e^{-t^2/2} dt.$$

$$= a_x^2 + \frac{1}{\sqrt{2\pi}} \sigma_x^2 \int_{-\infty}^{\infty} t (t e^{-t^2/2}) dt + 2\sigma_x a_x [0]$$

$$= a_x^2 + \frac{\sigma_x^2}{\sqrt{2}}$$

The mean of gaussian R.V equals to a_x and varience of gaussian R.V equals to σ_x^2.

Uniform density function:- we know that $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{else} \end{cases}$

Mean:- $E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a} = \left.\frac{x^2}{2(b-a)}\right|_a^b = \frac{a+b}{2}.$

Varience:-

$var(x) = \sigma_x^2 = E[x^2] - E[x].$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \frac{1}{b-a} dx = \left.\frac{x^3}{3(b-a)}\right|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{\cancel{(b-a)}(b^2 + ab + a^2)}{3\cancel{(b-a)}} = \frac{a^2 + ab + b^2}{3}$$

$$var(x) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+b^2+2ab}{4} = \frac{(b-a)^2}{12}$$

Skew:-

skew of the R.V is given by $\mu_3 = E[x^3] - 3\bar{x}\sigma_x^2 - \bar{x}^3$

$$E[x^3] = \int_a^b x^3 \frac{1}{b-a} dx = \left.\frac{x^4}{4(b-a)}\right|_a^b = \frac{1}{4\cancel{(b-a)}} \cancel{(b-a)} (b+a)(b^2+a^2)$$

$$= (b+a)(b^2+a^2)/4$$

$$\mu_3 = E[x^3] - 3\bar{x}\sigma_x^2 - \bar{x}^3$$

$$= \frac{(b+a)(b^2+a^2)}{4} - 3\left(\frac{a+b}{2}\right)\frac{(b-a)^4}{144} - \left(\frac{a+b}{2}\right)^3$$

$$= \frac{(b+a)(b^2+a^2)}{4} - 3\frac{(b+a)(b-a)^4}{288} - \frac{(a+b)^3}{8}$$

$$= 0$$

skewness :: of the density function is defined as $\frac{\mu_3}{\sigma_x^3} = 0$.

→ Exponential Probability density function ::

we know that the exponential prob density function is $f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x \geq a \\ 0 & x < a \end{cases}$

Mean :: $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^{\infty} x \cdot \frac{1}{b} e^{-(x-a)/b}\,dx$

$$= \frac{1}{b}\left[x e^{-(x-a)/b}(-b) + \int b e^{-(x-a)/b}\,dx\right]$$

$$= \frac{1}{b}\left(-bx e^{-(x-a)/b} - b^2 e^{-(x-a)/b}\right)\Big|_a^{\infty} = a+b.$$

varience :: $var(X) = \sigma_x^2 = E[x^2] - \bar{x}^2$

$$E\{x^2\} = \frac{1}{b}\int_{-\infty}^{\infty} x^2 e^{-(x-a)/b}$$

$$= \frac{1}{b}\left[x^2 e^{-(x-a)/b}(-b) + \int b e^{-(x-a)/b} 2x\right].$$

$$= \frac{1}{b}\left(-bx^2 e^{-(x-a)/b} + 2b\int x e^{-(x-a)/b}\right).$$

$$= \frac{1}{b}\left\{\left[-bx^2 e^{-(x-a)/b} + 2b\left[-bx e^{-(x-a)/b} + \int b e^{-(x-a)/b}\right].\right\}\right.$$

$$= \frac{1}{b}\left(-bx^2 e^{-(x-a)/b} + 2b\left[-\cancel{x}b e^{-(x-a)/b} + \int b e^{-(x-a)/b}\right]\right].$$

$$= \frac{1}{b}\left\{-bx^2 e^{-(x-a)/b} + 2b\left[-bx e^{-(x-a)/b} - b^2 e^{-(x-a)/b}\right)\right).$$

$$= -x^2 e^{-(x-a)/b} - 2bx e^{-(x-a)/b} - 2b^2 e^{-(x-a)/b}\Big|_a^{\infty}$$

$$= 2b^2 + 2ab + a^2$$

$$var(x) = 2b^2 + 2ab + a^2 - a^2 - 2ab - b^2 = b^2.$$

skew :: $\mu_3 = E[x^3] - 3\bar{x}\sigma_x^2 + \bar{x}^3$

$$\int x^3 e^{ax}\,dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right].$$

$$E[x^3] = \frac{1}{b}\int_a^{\infty} x^3 e^{-(x-a)/b}\,dx$$

$$= \frac{1}{b} e^{-(x-a)/b}\left[x^3(-b) - 3x^2 b^2 + 6x(-b^3) - 6b^4\right].$$

$= \frac{1}{b} e^{-(x-a)/b} \left[-bx^3 - 3x^2b^2 - 6b^3x - 6b^4\right]_a^{\infty}$

$\therefore \frac{1}{b}\left[a^3b + 3a^2b^2 + 6b^3a + 6b^4\right] = a^3 + 3a^2b + 6b^2a + 6b^3.$

$\mu_3 = a^3 + 3a^2b + 6ab^2 + 6b^3 - 3(a+b)(b^2) - (a+b)^3$

$= a^3 + \cancel{3a^2b} + \cancel{6ab^2} + 6b^3 - \cancel{3ab^2} - 3b^3 - a^3 - b^3 - \cancel{3a^2b} - \cancel{3ab^2}$

$= 2b^3$

skewness: $\frac{\mu_3}{\sigma_x^3} = \frac{2b^3}{b^3} = 2$

Binomial density function:

we know that $f_X(x) = \sum_{k=0}^{N} ({}^N C_k)\, p^k q^{N-k}\, \delta(x-k)$ —(1)

where $x = 0, 1, 2, \ldots N$

now $P(x) = {}^N C_x\, p^x q^{N-x}$ where $x = 0, 1, 2, \ldots N.$

Mean: w.k.T $E[x] = \sum_{i=1}^{N} x_i P(x_i)$ then $E[x] = \sum_{i=0}^{N} x P(x).$

$E[x] = \sum_{x=0}^{N} x \cdot {}^N C_x\, p^x q^{N-x} = \sum_{x=1}^{N} x \cdot ({}^N C_x)\, p^x q^{N-x}.$

since the first term on the summation is zero.

$= \sum_{x=1}^{N} x \cdot \frac{N!}{x!(N-x)!} p^x q^{N-x}$

$= \sum_{x=1}^{N} \frac{x \cdot N(N-1)!}{x(x-1)!\,[(N-1)-(x-1)]!}\, p\, p^{x-1} q^{[(N-1)-(x-1)]}$

$= Np \sum_{x=1}^{N} {}^{(N-1)}C_{(x-1)}\, p^{x-1} q^{[(N-1)-(x-1)]}$

= let $x-1 = j \Rightarrow x=1, j=0$

$x = N \Rightarrow j = N-1$

$E[x] = Np \sum_{j=0}^{(N-1)} {}^{(N-1)}C_j\, p^j q^{(N-1)-j}$

$= Np(p+q)^{N-1}$ $\quad \left[\because \sum_{x=0}^{N} {}^N C_x\, p^x q^{N-x} = (p+q)^N\right].$

$= Np(1)^{N-1} = Np$

variance: $E[x^2] - \bar{x}^2$

mean square value $E[x^2] = \sum_{x=0}^{N} x^2 P(x).$

$$= \sum_{x=0}^{N} x^2 N_{C_x} p^x . q^{N-x}$$

$$= \sum_{x=0}^{N} [x(x-1)+x] N_{C_x} p^x q^{N-x}.$$

$$= \sum_{x=0}^{N} x(x-1) N_{C_x} p^x q^{N-x} + \sum_{x=0}^{N} x . N_{C_x} p^x q^{N-x}$$

$$= \sum_{x=2}^{N} x(x-1) N_{C_x} p^x q^{N-x} + \sum_{x=1}^{N} x . N_{C_x} p^x q^{N-x}$$

$$= \sum_{x=2}^{N} x(x-1) . \frac{N(N-1)(N-2)!}{x(x-1)(x-2)! [(N-2)-(x-2)]!} p^{x-2} p^2 q^{[(N-2)-(x-2)]} + Np.$$

$$= N.(N-1) p^2 \sum_{x=2}^{N} (N-2)_{C_{(x-2)}} p^{(x-2)} q^{[(N-2)-(x-2)]} + NP$$

$$= N(N-1)p^2 (p+q)^{N-2} + Np \quad = \cancel{N(N-1)p^2 + Np(1)^{N-1}}$$

$$= \cancel{Np[N-1+}.$$

$$= N(N-1)p^2 + Np$$

$$= Np[\cancel{Np}(N-1)p+1] = N(N-1)p^2 + Np.$$

$$var(x) = E[x^2] - \bar{x}^2 = N(N-1)p^2 + Np - N^2p^2$$

$$= N^2p^2 - Np^2 + Np - \cancel{N^2p^2} = Np - Np^2 = Np[1-p] = Npq$$

$$\boxed{\sigma_x^2 = Npq}$$

Poission's density function:- we know that $f_x(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$ —(1)

when $x = 0, 1, 2, \ldots \infty$

now $p(x) = e^{-b} \frac{b^{x}}{x!}$ $\quad x = 0, 1, 2, \ldots \infty.$

$$E[x] = \sum_{x=0}^{\infty} x.p(x) = \sum_{x=0}^{\infty} x e^{-b} \frac{b^x}{x!} = e^{-b} \sum_{x=0}^{\infty} x . \frac{b^x}{x!}$$

since the 1st term in summation is zero for $x=0$

$$E[x] = e^{-b} \sum_{x=1}^{\infty} \frac{x . b^{x-1} . b^1}{x(x-1)!} = b e^{-b} \sum_{x=1}^{\infty} \frac{b^{x-1}}{(x-1)!}$$

$$= b e^{-b} [1 + b + \frac{b^2}{2!} + \frac{b^3}{3!} + \ldots] = b e^{-b} e^{b} = b.$$

$$E[x] = b \quad (\bar{x})$$

varience :- $\sigma_x^2 = E[x^2] - \bar{x}^2$

$$E[x^2] = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \frac{e^{-b} b^x}{x!} = \sum_{x=0}^{\infty} [x(x-1)+x] \frac{e^{-b} b^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-b} b^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-b} b^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{x(x-1)\, e^{-b}\, b^{x-2}\, b^2}{x(x-1)(x-2)!} + \sum_{x=1}^{\infty} \frac{x \cdot e^{-b}\, b^{x-1} \cdot b}{x(x-1)!}$$

$$= b^2 \sum_{x=2}^{\infty} \frac{e^{-b}\, b^{x-2}}{(x-2)!} + b \sum_{x=1}^{\infty} \frac{e^{-b}\, b^{x-1}}{(x-1)!}$$

$$= b^2 (1) + b = b^2 + b$$

$\left[\because \sum_{x=1}^{\infty} \frac{e^{-b}\, b^{x-1}}{(x-1)!} = 1\right.$ it summation of all probs is unity].

$$\Rightarrow \sigma_x^2 = b^2 + b - b^2 = b$$

$$\Rightarrow \sigma_x^2 = b.$$

Functions that give moments:-

Two functions are generally used to calculate the mth moments of a R.V "x". They are 1. characteristic function

2. Moment generating function (MGF).

characteristic function:-

characteristic function of a random variable "x" is defined as

$$\phi_x(\omega) = E[e^{j\omega x}] \quad -①$$

where $j = \sqrt{-1}$ and this is a function of real variable and $-\infty < \omega < \infty$.

Now $\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} f_x(x)\, e^{j\omega x}\, dx$

$$= \int_{-\infty}^{\infty} f_x(x)\, e^{j\omega x}\, dx \rightarrow ②$$

This equation states that $\phi_x(\omega)$ is a Fourier transform of $f_x(x)$. [with the sign of ω is reversed].

Now we can calculate $f_x(x)$ by knowing $\phi_x(\omega)$ from inverse fourier transform [with sign of x is reversed].

$$f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(\omega)\, e^{-j\omega x}\, d\omega \quad -③.$$

The characteristic function of a random variable "x" is $\phi_x(\omega)$. Then the mth moment of x is given by

$$m_n = (-j)^n \left.\frac{d^n \phi_x(\omega)}{d\omega^n}\right|_{\omega=0} \quad -④$$

Theorem: If $\phi_x(\omega)$ is a characteristic function of R.V "x" then the mth moment is given by $m_n = (-j)^n \left.\frac{d^n \phi_x(\omega)}{d\omega^n}\right|_{\omega=0}$.

Proof: The characteristic function of a R.V x is

$$\phi_x(\omega) = E[e^{j\omega x}].$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

consider $\frac{d^n \phi_X(\omega)}{d\omega^n} = \frac{d^n}{d\omega^n} \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx.$

$$= \int_{-\infty}^{\infty} \frac{d^n}{d\omega^n} (e^{j\omega x}) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} (jx)^n f_X(x)\,dx \qquad \left[\because \frac{d^n}{d\omega^n}(e^{j\omega x}) = e^{j\omega x} \cdot (jx)^n\right]$$

$$\frac{d^n \phi_X(\omega)}{d\omega^n} = j^n \int_{-\infty}^{\infty} x^n e^{j\omega x} f_X(x)\,dx$$

$$\Rightarrow \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0} = j^n \int_{-\infty}^{\infty} x^n f_X(x)\,dx. = j^n E[X^n].$$

$$\left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0} = j^n E[X^n] = j^n m_n$$

$$m_n = \left(\frac{1}{j}\right)^n \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0} = (-j)^n \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

Properties:- 1. characteristic function is unity at $\omega=0$ and is given by

$$\phi_X(\omega)\big|_{\omega=0} = \phi_X(0) = 1$$

proof.. The characteristic function of R.V is $\phi_X(\omega) = E[e^{j\omega X}]$.

$$\phi_X(\omega)\big|_{\omega=0} = E[1] = 1$$

2. The magnitude of characteristic function is unity at $\omega=0$ ie

$$|\phi_X(\omega)| \le \phi_X(0) = 1$$

proof:- $\phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x).\,dx$

$$|\phi_X(\omega)| = \left|\int_{-\infty}^{\infty} e^{j\omega x}. f_X(x)\,dx\right|$$

$$|\phi_X(\omega)| \le \int_{-\infty}^{\infty} |e^{j\omega x}|\,|f_X(x)|\,dx \qquad [\because |xy| \le |x||y|].$$

$$|\phi_X(\omega)| \le \int_{-\infty}^{\infty} 1.|f_X(x)|\,dx \qquad [\because |e^{j\omega x}| = 1].$$

$$|\phi_X(\omega)| \le \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$|\phi_X(\omega)| \le 1 \text{ (or) } |\phi_X(\omega)| \le \phi_X(0) = 1$$

3.

3. $\phi_x(\omega)$ and $\phi(-\omega)$ are conjugate symmetry ie $\phi_x(\omega) = \phi^*(\omega)$ or $\phi^*(-\omega) = \phi_x(\omega)$ (29)

$\phi_x(-\omega) = \phi^*(\omega)$

Proof: we know that $\phi_x(\omega) = E[e^{j\omega x}]$. (or)

$\phi_x(-\omega) = E[e^{j(-\omega)x}]$

$= E[e^{-j\omega x}] = E[e^{j\omega x}]^*$

$= \phi^*(\omega)$

$\phi_x(\omega) = E[e^{j\omega x}]$.

$\phi_x(-\omega) = E[e^{-j\omega x}]$.

$\phi_x^*(-\omega) = E[e^{j\omega x}]$

$= \phi_x(\omega)$.

Property 4:

If $\phi_x(\omega)$ is a characteristic function of R.V "x" then ch-ic function of R.V $y = ax+b$ is given by

$\phi_y(\omega) = e^{j\omega b}\phi_x(a\omega)$ where a, b are real constants).

Proof: $\phi_y(\omega) = E[e^{j\omega y}]$

$= E[e^{j\omega[ax+b]}] = E[e^{j(a\omega)x + j\omega b}]$

$= E[e^{j\omega b}. e^{ja\omega x}] = e^{j\omega b}. E[e^{ja\omega x}]$

$= e^{j\omega b} E[e^{j(a\omega)x}] = e^{j\omega b}.\phi_x(a\omega)$.

Property 5:

If $\phi_x(\omega)$ is a ch-ic function of R.V x then

$\phi_x(c\omega) = \phi_{cx}(\omega)$ where c is real constant.

Proof: $\phi_x(c\omega) = E[e^{jc\omega x}]$

we know that $\phi_x(\omega) = E[e^{j\omega x}]$.

$\Rightarrow \phi_x[(c\omega)] = E[e^{j\omega(cx)}] = \phi_{cx}(\omega)$.

Property 6:

If x and y are two independent R.Vs then $\phi_{x+y}(\omega) = \phi_x(\omega).\phi_y(\omega)$.

Proof: we know that the characteristic function of R.V is defined as

$\phi_x(\omega) = E[e^{j\omega x}]$.

$\phi_{x+y}(\omega) = E[e^{j\omega(x+y)}]$

$= E[e^{j\omega x + j\omega y}]$

$= E[e^{j\omega x}. e^{j\omega y}]$.

x and y are two independent random variables

ie

$\phi_{x+y}(\omega) = \phi_x(\omega).\phi_y(\omega)$

→ The characteristic function for a Random variable x is given by: $Q_x(\omega) = \frac{1}{(1-j2\omega)^{N/2}}$. Find (i) Mean & second moments of 'x'.

sol: given that the characteristic function of a R.V x is given by

$$Q_x(\omega) = \frac{1}{(1-j2\omega)^{N/2}}$$

W.K.T nth moment $m_n = (-j)^n \frac{d^n \phi_x(\omega)}{d\omega^n}\Big|_{\omega=0}$

Mean / 1st moment / $E(x)$

$$m_1 = (-j)^1 \frac{d\phi_x(\omega)}{d\omega}\Big|_{\omega=0} = -j\frac{d}{d\omega}\left[\frac{1}{(1-2j\omega)^{N/2}}\right]\Big|_{\omega=0}$$

$$= (-j)\frac{d}{d\omega}(1-2j\omega)^{-N/2}\Big|_{\omega=0}$$

$$= (-j)(-N/2)(1-2j\omega)^{-\frac{N}{2}-1}(-2j)\Big|_{\omega=0}$$

$$= (-j)\cdot -\frac{N}{2}(-2j) = -Nj^2 = N$$

$m_1 = N$

2nd moment: $m_2 = (-j)^2 \frac{d^2 Q_x(\omega)}{d\omega^2}\Big|_{\omega=0} = (-j)^2\frac{d^2}{d\omega^2}(1-2j\omega)^{-N/2}\Big|_{\omega=0}$

$$= (-j)^2\frac{d}{d\omega}\left[-\frac{N}{2}[1-2j\omega]^{-\frac{N}{2}-1}(-2j)\right]\Big|_{\omega=0}$$

$$= (-j)^2\frac{d}{d\omega}\left[Nj(1-2j\omega)^{-\frac{N}{2}-1}\right]\Big|_{\omega=0}$$

$$= -1\left[Nj\left(-\frac{N}{2}-1\right)(1-2j\omega)^{-\frac{N}{2}-2}(-2j)\right]\Big|_{\omega=0}$$

$$= -1\left[2Nj^2\left(\frac{N}{2}+1\right)(1-2j\omega)^{-\frac{N}{2}-\cancel{2}}\right]_{\omega=0}$$

$$= -1\left[-2N\left(\frac{N+2}{2}\right)\right] = \frac{\cancel{2}N(N+2)}{\cancel{2}} = N(N+2).$$

variance $= N^2+2N-N = N^2+N$.

NOTE:- By using characteristic function we can also find the variance.

→ Find the density function of a random variable "x" if the characteristic function $\phi_x(\omega) = \begin{cases} 1-|\omega| & |\omega|\le 1 \\ 0 & \text{else} \end{cases}$

$|\omega| \le 1$
$\omega \le 1$ or $-\omega \le 1$
$\omega \le 1$ or $\omega \ge -1$
$-1 \le \omega \le 1$

sol: $$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_x(\omega)e^{-j\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1-|\omega|\,e^{-j\omega x}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} 1-|\omega|\,e^{-j\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{0}(1+\omega)e^{-j\omega x}\,d\omega + \frac{1}{2\pi}\int_{0}^{1}(1-\omega)e^{-j\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{0} e^{-j\omega x}\, d\omega + \frac{1}{2\pi}\int_{-1}^{0} \omega e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\cdot \frac{e^{-j\omega x}}{-jx}\Big|_{-1}^{0} + \frac{1}{2\pi}\left[\omega\cdot\frac{e^{-j\omega x}}{-jx} - \int_{-1}^{0}\frac{e^{-j\omega x}}{-jx}\cdot 1\right]$$

$$= \frac{-1}{2\pi jx} + \frac{1}{2\pi jx}e^{jx} + \frac{1}{2\pi}\left[-\omega\frac{e^{-j\omega x}}{jx} + \frac{1}{jx}\left[\frac{e^{-j\omega x}}{-jx}\right]_{-1}^{0}\right]$$

$$= -\frac{1}{2\pi jx} + \frac{1}{2\pi jx}e^{jx} + \frac{1}{2\pi}\left[0 + \frac{1}{x^2} - \frac{e^{jx}}{jx} + \frac{1}{x^2}e^{jx}\right]$$

$$= -\frac{1}{2\pi jx} + \cancel{\frac{1}{2\pi jx}e^{jx}} + \frac{1}{2\pi x^2} - \cancel{\frac{1}{2\pi jx}e^{jx}} + \frac{1}{2\pi x^2}e^{jx}$$

$$= \frac{-1}{2\pi jx} + \frac{1}{2\pi x^2} + \frac{1}{2\pi x^2}e^{jx}$$

$$\rightarrow \frac{1}{2\pi}\int_0^1 (1-\omega)e^{-j\omega x}\, d\omega = \frac{1}{2\pi}\int_0^1 e^{-j\omega x}\, d\omega - \frac{1}{2\pi}\int_0^1 \omega e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega x}}{-jx}\right|_0^1 - \frac{1}{2\pi}\left[\omega\cdot\frac{e^{-j\omega x}}{-jx} + \int \frac{e^{-j\omega x}}{-jx}\right]\Bigg|_0^1$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega x}}{-jx}\right|_0^1 - \frac{1}{2\pi}\left[-\frac{\omega e^{-j\omega x}}{jx} + \frac{1}{jx}\left[\frac{e^{-j\omega x}}{-jx}\right]\right]_0^1$$

$$= -\frac{1}{2\pi jx}e^{-jx} + \frac{1}{2\pi jx} - \frac{1}{2\pi}\left[-\frac{e^{-jx}}{jx} + \frac{1}{x^2}e^{-jx} + 0 - \frac{1}{x^2}\right]$$

$$= -\cancel{\frac{1}{2\pi jx}e^{-jx}} + \frac{1}{2\pi jx} + \cancel{\frac{1}{2\pi jx}e^{-jx}} - \frac{1}{2\pi x^2}e^{-jx} + \frac{1}{2\pi x^2}$$

$$= \frac{1}{2\pi jx} - \frac{1}{2\pi x^2}e^{jx} + \frac{1}{2\pi x^2}$$

$$= -\cancel{\frac{1}{2\pi jx}} + \frac{1}{2\pi x^2} + \cancel{\frac{1}{2\pi x^2}e^{jx}} + \cancel{\frac{1}{2\pi jx}} - \cancel{\frac{1}{2\pi x^2}e^{-jx}} + \frac{1}{2\pi x^2}$$

$$f_X(x) = \frac{1}{2\pi x^2} + \frac{1}{2\pi x^2}e^{jx} - \frac{1}{2\pi x^2}e^{-jx} + \frac{1}{2\pi x^2} = \frac{1}{2\pi x^2}\left[2-(e^{jx}+e^{-jx})\right] = \frac{1}{\pi x^2}\left(1 - \frac{e^{jx}+e^{-jx}}{2}\right)$$

$$= \frac{1}{\pi x^2}\left[1-\cos x\right]$$

→ Find the characteristic function of a exponential density function of a R-V X and also find its first moment.

Sol:- we know that $f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & \text{for } x \geq a \\ 0 & \text{elsewhere} \end{cases}$

The characteristic function is $\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx.$

$$= \int_{a}^{\infty} \frac{1}{b} e^{-(x-a)/b} e^{j\omega x}\, dx = \frac{1}{b}\int_{a}^{\infty} e^{-\frac{x}{b} + \frac{a}{b} + j\omega x}\, dx.$$

$$= \frac{1}{b}\int_{a}^{\infty} e^{-x\left(\frac{1}{b} - j\omega\right) + \frac{a}{b}}\, dx = \frac{1}{b}\left| \frac{e^{-x\left(\frac{1}{b}-j\omega\right)+\frac{a}{b}}}{-\left(\frac{1}{b}-j\omega\right)} \right|_{a}^{\infty}$$

$$= \frac{1}{b}\left[\frac{e^{-\frac{a}{b} + aj\omega + \frac{a}{b}}}{\frac{1}{b} - j\omega} \right]$$

$$\phi_X(\omega) = \frac{e^{aj\omega}}{(1 - bj\omega)}$$

First moment:-

→ Find the characteristic function of a uniform distributed R.V 'X'.

sol:- we know that uniform density function $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else,} \end{cases}$

The characteristic function $\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx.$

$$= \int_{a}^{b} \frac{1}{b-a} e^{j\omega x}\, dx = \frac{1}{b-a} \left.\frac{e^{j\omega x}}{j\omega}\right|_a^b = \frac{1}{(b-a)j\omega}\left\{ e^{j\omega b} - e^{j\omega a} \right\}.$$

→ show that the ch-ic function of a R.V having binomial density is $\phi_X(\omega) = \left[1 - p + pe^{j\omega}\right]^N$

sol: we know that the binomial density function is

$$P(x) = {}^N C_x\, p^x (1-p)^{N-x} \qquad x = 0, 1, 2 \cdots N$$

characteristic function $\phi_X(\omega) = E\left[e^{j\omega X}\right]$

$$= \sum_{x=0}^{N} e^{j\omega x} P(x) = \sum_{x=0}^{N} e^{j\omega x}\, {}^N C_x\, p^x (1-p)^{N-x}$$

$$= \sum_{x=0}^{N} {}^N C_x \left(pe^{j\omega}\right)^x (1-p)^{N-x} \qquad \left[\because \sum_{x=0}^{N} {}^N C_x\, p^x q^{N-x} = (p+q)^N\right].$$

$$= \left(pe^{j\omega} + (1-p)\right)^N$$

$$= \left[(1-p) + pe^{j\omega}\right]^N$$

→ show that the characteristic function of a R.V having Poisson density is $\phi_x(\omega) = exp[-b(1-e^{j\omega})]$.

sol: w.k.T Poisson density function is

$$P(x) = \frac{e^{-b} b^x}{x!} \quad x = 0,1,2 \ldots \infty.$$

characteristic function $\phi_x(\omega) = E[e^{j\omega x}]$.

$$= \sum_{x=0}^{\infty} e^{j\omega x} P(x) = \sum_{x=0}^{\infty} e^{j\omega x} . \frac{e^{-b} . b^x}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} \frac{e^{j\omega x} . b^x}{x!} = e^{-b} \sum_{x=0}^{\infty} \frac{(be^{j\omega})^x}{x!}$$

$$= e^{-b}\left[1 + \frac{be^{j\omega}}{1!} + \frac{(be^{j\omega})^2}{2!} + \ldots\right].$$

$$= e^{-b} . e^{be^{j\omega}}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$$

$$= exp(-b + be^{j\omega})$$

$$= exp[-b[1-e^{j\omega}]]$$

→ show that ch-ic function of a random variable "x" of a poisson distributed R.V is $exp(-\lambda(1-e^{j\omega}))$.

sol: w.k.t on Poisson distribution

$$P(x) = \frac{e^{-b} b^x}{x!} \quad x = 0,1,2 \ldots \infty$$

ch-ic function $\phi_x(\omega) = E[e^{j\omega x}]$

$$\phi_x(\omega) = \sum_{x=0}^{\infty} P(x) e^{j\omega x} = \sum_{x=0}^{\infty} e^{j\omega x} . \frac{e^{-b} b^x}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} \frac{e^{j\omega x} b^x}{x!} = e^{-b} \sum_{x=0}^{\infty} \frac{(be^{j\omega})^x}{x!} = exp[-b(1-e^{j\omega})].$$

Moment Generating function:-

Moment generating function of a random variable "x" is defined as

$$M_x(v) = E[e^{vx}] \quad \text{—①}.$$

where "v" is a real variable $-\infty < v < +\infty$.

and it is given by $M_x(v) = E[e^{vx}]$

$$M_x(v) = \int_{-\infty}^{\infty} e^{vx} f_x(x) dx \quad \text{—②}.$$

m^{th} moments given by the moment generating function is

$$m_n = \frac{d^n}{du^n} M_x(u) \Big|_{u=0} \quad -(3)$$

Theorem:- $M_x(u)$ is a moment generating function of a R.V "x" then nth moments $m_n = \frac{d^n}{du^n} M_x(u)$

Proof:- Moment generating function of a R.V is given by

$$M_x(u) = E[e^{ux}]$$

$$= E\left[1 + \frac{ux}{1!} + \frac{(ux)^2}{2!} + \frac{(ux)^3}{3!} + \cdots + \frac{(ux)^n}{n!}\right].$$

$$= E[1] + uE[x] + \frac{u^2}{2!}E[x^2] + \frac{u^3}{3!}E[x^3] + \cdots + \frac{u^n}{n!}E[x^n] +$$

$$= 1 + u m_1 + \frac{u^2}{2!} m_2 + \frac{u^3}{3!} m_3 + \frac{u^4}{4!} m_4 + \cdots + \frac{u^n}{n!} m_n + \cdots$$

$$\left\{ m_n = \frac{d^n}{du^n} M_x(u) \Big|_{u=0} \right\}$$

Differentiating above equation w.r.t "u"

$$= \frac{d}{du} M_x(u) = m_1 + \frac{m_2}{2}(2u) + \frac{m_3}{6} 3u^2 + \cdots + \frac{m_n}{n!} n u^{n-1} + \cdots$$

$$= m_1 + m_2(u) + \frac{m_3}{2} u^2 + \cdots \frac{m_n}{(n-1)!} u^{n-1} + \cdots = m_1$$

$$\boxed{m_1 = \frac{d}{du} M_x(u)}$$

consider $\frac{d^2 M_x(u)}{du^2} = m_2 + \frac{m_3}{2}(2u) + \cdots$

$$= m_2 + m_3 u$$

$$\frac{d^2 m_x(u)}{du^2} = m_2$$

similarly $\boxed{\frac{d^n M_x(u)}{du^n} = m_n}$.

Properties:-

1. Moment generating function is unity at $u=0$ & is given by

$$M_x(u)\Big|_{u=0} = M_x(0) = 1$$

w.k.T $M_x(u) = E[e^{ux}]$

$$M_x(u)\Big|_{u=0} = E[1] = 1.$$

2. If $M_x(u)$ is a moment generating function of a random variable "x" then M.G.F of $y = ax+b$. is given by $M_y(u) = e^{ub} M_x(au)$ where a, b are real constant.

Proof:- we know that $M_x(u) = E[e^{ux}]$

$$M_y(u) = E[e^{uy}] = E[e^{u(ax+b)}] = E[e^{(au)x} \cdot e^{ub}].$$
$$= e^{ub} \cdot E[e^{(au)x}] = e^{ub} M_x(au).$$

3. If $M_x(u)$ is a moment generating function of R.V "x" then $M_x(cu) = M_{cx}(u)$ where c is a real constant.

Proof:- w.k.T $M_x(u) = E(e^{ux})$

$$M_x(cu) = E[e^{cux}] = E[e^{u(cx)}] = M_{cx}(u).$$

4. If x and y are two independent random variables with moment generating functions $M_x(u)$ and $M_y(u)$ then $M_{x+y}(u) = M_x(u) \cdot M_y(u)$

Proof:- w.k.T $M_x(u) = E[e^{ux}]$

$$M_{x+y}(u) = E[e^{u(x+y)}] = E[e^{ux} \cdot e^{uy}] = E[e^{ux}] \cdot E[e^{uy}]$$
$$M_{x+y}(u) = M_x(u) \cdot M_y(u).$$

5. If $M_x(u)$ is a moment generating function of R.V "x" then Moment generating function of $y = \frac{x+a}{b}$ is given by $M_y(u) = e^{au/b} M_x(u/b)$.

Proof:- w.k.T $M_x(u) = E[e^{ux}]$.

$$M_y(u) = E[e^{u(\frac{x+a}{b})}] = E[e^{ux/b} \cdot e^{au/b}].$$
$$= e^{au/b}[E[e^{ux/b}]] = e^{au/b} \cdot M_x(u/b).$$

Problems

→ Find the moment generating function (MGF) for exponential distributed R.V and also find mean value

Sol:- we know that exponential density function is

$$f_x(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & \text{for } x \geq a \\ 0 & \text{else} \end{cases}$$

where a, b are real consts, $-\infty < a < \infty$, $b > 0$

Moment generating function of a "x" is defined as

$$M_x(u) = E[e^{ux}] = \int_{-\infty}^{\infty} e^{ux} \cdot f_x(x)\, dx = \int_{-\infty}^{\infty} e^{ux} \cdot \frac{1}{b} e^{-(x-a)/b} dx$$

$$= \frac{1}{b} \int_a^{\infty} e^{ux} e^{-\frac{x}{b}} e^{a/b}\, dx$$

$$= \frac{1}{b} e^{a/b} \int_a^\infty e^{ux} \cdot e^{-x/b} dx = \frac{1}{b} e^{a/b} \int_a^\infty e^{-x(-u+\frac{1}{b})} dx$$

$$= \frac{1}{b} e^{a/b} \left[\frac{e^{-x(\frac{1}{b}-u)}}{-(\frac{1}{b}-u)} \right]_a^\infty = \frac{1}{b} e^{a/b} \frac{-1}{\frac{1}{b}-u} \left[0 - e^{-a(\frac{1}{b}-u)} \right].$$

$$= \frac{1}{b} e^{a/b} \left[\frac{-b}{1-bu} \right] \left[\cancel{e^{-a/b}} \cdot e^{+au} \right]$$

$$\boxed{M_x(u) = \frac{e^{av}}{1-ub}}$$

we know that nth moment

$$\frac{d^n}{du^n} M_x(u) \Big|_{v=0}$$

mean value $m_1 = \frac{d}{dv} m_x(v) \Big|_{v=0} = \frac{d}{dv} \left[\frac{e^{av}}{1-vb} \right] \Big|_{v=0}$

$$= \frac{(1-vb) e^{av} \cdot a - e^{ab} (-b)}{(1-ub)^2} \Big|_{v=0} = \frac{e^{au} [a - vab + b]}{1-ub} \Big|_{v=0}$$

$$= a + b$$

→ uniform density function

w.k.T " " " $f_x(x) = \frac{1}{b-a}$ $a \le x \le b$

$$M_x(u) = E[e^{ux}] = \int_{-\infty}^{\infty} e^{ux} f_x(x) dx = \int_a^b e^{ux} \cdot \frac{1}{b-a} dx.$$

$$= \frac{1}{b-a} \frac{e^{ux}}{u} \Big|_a^b = \frac{1}{(b-a)u} \left[e^{ub} - e^{ua} \right] = \frac{e^{ub} - e^{ua}}{u(b-a)}$$

Mean value :

w.k.T mean value $= \frac{d}{du} M_x(u) = \frac{d}{du} \frac{e^{ub} - e^{ua}}{u(b-a)}$

$$= \frac{1}{b-a} \left[\frac{v \cdot e^{ub}(b) - v e^{ua}(a) - e^{ub} + e^{ua}}{u^2} \right] \Big|_{v=0}$$

$$= 0.$$

→ Binomial density function

$$P(x) = {}^N C_x \, p^x q^{N-x} \qquad x = 0, 1, 2 \cdots N$$

$$M_x(u) = E[e^{ux}]$$

$$= \sum_{x=0}^{N} e^{ux} \, {}^N C_x \, p^x q^{N-x}$$

$$= \sum_{x=0}^{N} {}^N C_x (pe^u)^x q^{N-x}$$

w.k.t $\sum_{x=0}^{n} {}^{N}C_x \, p^x q^{N-x} = (p+q)^N$

$\therefore \; M_x(v) = (pe^v + q)^N$ (or)

$M_x(v) = \{pe^v + (1-p)\}^N$

NOTE :: ch-ic function can be easily obtained by substituting $v = j\omega$ in MGF

i.e $\phi(\omega) = M_x(v)\big|_{v=j\omega}$

→

Poisson density function :-

$P(x) = \dfrac{e^{-b} \cdot b^x}{x!}$ $\quad x = 0, 1, 2 \cdots \infty$

$M_x(v) = E\{e^{vx}\} = \sum_{x=0}^{\infty} e^{vx} \cdot p(x) = \sum_{x=0}^{\infty} e^{vx} \dfrac{e^{-b} \, b^x}{x!}$

$= e^{-b} \sum_{x=0}^{\infty} \dfrac{(be^v)^x}{x!} = e^{-b}\left[1 + \dfrac{be^v}{1!} + \dfrac{(be^v)^2}{2!} + \cdots\right]$

$= e^{-b} \cdot e^{be^v} = e^{-b + be^v} = \exp(-b + be^v)$

$= \operatorname{Exp}[-b\{1 - e^v\}]$.

(or)

consider poisson distribution $P(x) = \dfrac{e^{-d} \cdot d^x}{x!}$, $x = 0, 1, 2 \cdots \infty$.

$M_x(v) = E\{e^{vx}\} = \sum_{x=0}^{\infty} e^{vx} \cdot \dfrac{e^{-d} \cdot d^x}{x!} = e^{-d} \sum_{x=0}^{\infty} \dfrac{(de^v)^x}{x!}$

$= e^{-d}\left[1 + \dfrac{de^v}{1!} + \dfrac{(de^v)^2}{2!} + \cdots\right] = \exp(-d(1 - e^v))$.

→ If the Random variable has the MGF $M_x(v) = \dfrac{2}{2-t}$ determine the variance of "x".

sol: Given MGF $= M_x(v) = \dfrac{2}{2-t}$

nth moments $m_n = \dfrac{d^n}{dt^n} M_x(t)\Big|_{t=0}$.

1st Moment :: $\dfrac{d}{dt} \dfrac{2}{2-t}\Big|_{t=0} = \dfrac{2}{(2-t)^2}\Big|_{t=0} = \dfrac{1}{2}$

2nd Moment :: $\dfrac{d}{dt}\left\{\dfrac{2}{(2-t)^2}\right\} = \dfrac{2(-2)(-1)}{(2-t)^3}\Big|_{t=0} = \dfrac{1}{2}$

variance $= m_2 - m_1^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

$\Rightarrow$ A R.V X has a p.d.f $f_X(x) = \frac{1}{2^x}$, $x = 1, 2, \dots n$. Find MGF

sol:.. Given $f_X(x) = \frac{1}{2^x}$ $x = 1, 2, \dots n$

Here X is a discrete R.V Hence $f_X(x) = P(x) = \frac{1}{2^x}$.

$$M_X(u) = E[e^{ux}] = \sum_{x=1}^{N} e^{ux} \cdot P(x) = \sum_{x=1}^{N} e^{ux} \cdot \frac{1}{2^x}$$

$$= \sum_{x=1}^{N} \left(\frac{e^u}{2}\right)^x = \sum_{x=0}^{n} \left(\frac{e^v}{2}\right)^x - \left(\frac{e^u}{2}\right)^0$$

we know that $\sum_{x=0}^{N} a^x = \frac{a^{N+1} - 1}{a - 1}$ here $a > 1$

$$\sum_{x=0}^{\infty} a^x = \frac{1}{1-a} \quad a < 1.$$

$$= \sum_{x=0}^{n} \left(\frac{e^v}{2}\right)^x - 1 = \frac{\left(\frac{e^v}{2}\right)^{n+1} - 1}{\frac{e^v}{2} - 1} - 1 = \frac{e^{u(n+1)} - 2^{n+1}}{(e^v - 2) 2^n} - 1$$

$$= \frac{e^{v(n+1)} - \cancel{2^{n+1}} - 2^n [e^v] + \cancel{2^{n+1}}}{(e^v - 2) 2^n}$$

$$= \frac{e^{v(n+1)} - 2^n e^u}{(e^v - 2) 2^n}.$$

$\Rightarrow$ The prob distribution of a R.V "X" is $P(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x$ $x = 0, 1, 2$

... so. Find MGF and also find first & 2nd moments.

Given $P(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x$

$$M_X(u) = E[e^{ux}] = \sum_{x=0}^{\infty} e^{ux} \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^x = \frac{2}{3} \sum_{x=0}^{\infty} \left(\frac{e^v}{3}\right)^x$$

$$= \frac{2}{3} \cdot \frac{1}{1 - e^v/3} = \frac{2}{3 - e^v}.$$

First moment:.. $\frac{d}{du} M_X(u) \Big|_{u=0} = \frac{d}{du}\left(\frac{2}{3-u}\right)\Big|_{v=0} = \frac{3 - e^u (0) - 2(-e^u)}{(3 - e^u)^2}\Big|_{u=0}$

$$= \frac{2e^v}{(3 - e^v)^2}\Big|_{v=0} = \frac{1}{2}.$$

second moment: $\frac{d}{dv}\left(\frac{2e^{v}}{(3-e^{v})^{2}}\right) = 2\frac{d}{dv}\frac{e^{v}}{(3-e^{v})^{2}}$

$$= \frac{2\cdot\left[(3-e^{v})^{2}e^{v} - e^{v}\,2(3-e^{v})(-e^{v})\right]}{(3-e^{v})^{4}}\Bigg|_{v=0}$$

$$= 2\cdot\frac{(3-e^{v})e^{v} + 2e^{2v}}{(3-e^{v})^{3}}\Bigg|_{v=0} = \frac{2[(3-1)+2]}{2^{3}} = 1$$

Transformations of a Random variable :-

Transformations are used to convert one random variable "x" into a new another random variable "y". It is denoted as $y = T(x)$ —①.

$$x \longrightarrow \boxed{y = T(x)} \longrightarrow y$$

$f_X(x)$ $\qquad\qquad$ $f_Y(y)$.

Transformation of a R.V "x" to a another R.V "y".

The random variable can be discrete, continuous or mixed R.V and the transformation "T" can be linear, non-linear, staircase segmented...

Here we are consider only 3 cases, depending on the form of "x" and "T."

1. Both x and y are continuous and "T" is either monotonically increasing or decreasing
2. Both x and y are continuous, and T is non-monotonic
3. x is discrete and "T" is continuous.

Note that the transformation in all three cases is assumed continuous.

Monotonic transformation of a continuous random variable

A Transformation "T" is called monotonically increasing if

if $T(x_1) < T(x_2)$ for any $x_1 < x_2$.

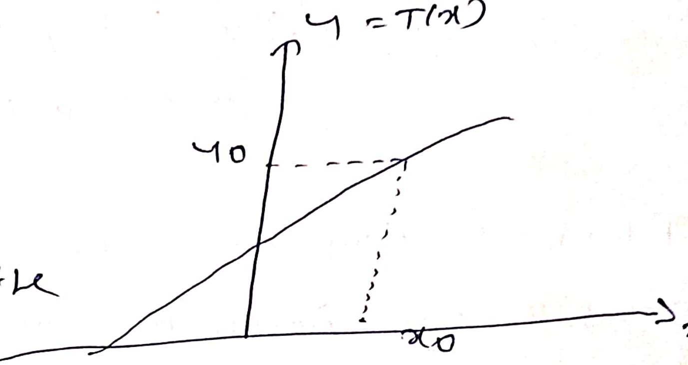
A Transformation "T" is called monotonically decreasing if

$T(x_1) > T(x_2)$ for any $x_1 < x_2$.

Monotonically increasing Transformation...

consider "T" is continuous and differentiable for all values of "x" for which $f_X(x) \neq 0$.

Let us consider another random variable "y" have a value of y_0 corresponding to x_0 of x as shown in fig.



From figure $y_0 = T(x_0)$

$$\Rightarrow x_0 = T^{-1}(y_0) \quad -(1)$$

Here T^{-1} represents the inverse of the Transformation "T". Now the prob of the event $\{Y \le y_0\}$ is equal to the prob of the event $\{X \le x_0\}$. because of one to one correspondance b/w X & Y.

$$\therefore P\{Y \le y_0\} = P\{X \le x_0\}$$

$$F_Y(y_0) = F_X(x_0).$$

$$= \int_{-\infty}^{y_0} f_Y(y)\, dy = \int_{-\infty}^{x_0} f_X(x) \cdot dx$$

$$= \int_{-\infty}^{y_0} f_Y(y)\, dy = \int_{-\infty}^{T^{-1}(y_0)} f_X(x)\, dx,$$

differentiating on both sides w.r.t y_0 using Leibnitz's rule

$$= \frac{d}{dy_0} \int_{-\infty}^{y_0} f_Y(y)\, dy = \frac{d}{dy_0} \int_{-\infty}^{T^{-1}(y_0)} f_X(x)\, dx$$

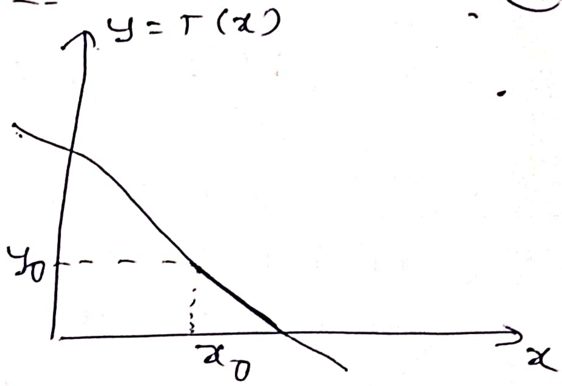
$$f_Y(y_0) = f_X(T^{-1}(y_0)) \frac{d}{dy_0} T^{-1}(y_0).$$

$$f_Y(y) = f_X[T^{-1}(y)] \frac{d\,T^{-1}(y)}{dy}$$

(or) $$\boxed{f_Y(y) = f_X(x) \cdot \frac{dx}{dy}}$$

Monotonically decreasing Transformation :- (35)

From figure



Monotonically decreasing Transformation.

$$P\{Y \le y_0\} = P\{X > x_0\}.$$

$$\Rightarrow P\{Y \le y_0\} = 1 - P\{X \le x_0\}.$$

$$\Rightarrow F_Y(y_0) = 1 - F_X(x_0).$$

$$\Rightarrow \int_{-\infty}^{y_0} f_Y(y)\,dy = 1 - \int_{-\infty}^{x_0} f_X(x)\,dx$$

differentiate on both sides w.r.t "y_0" using Leibniz's rule

$$\frac{d}{dy_0}\int_{-\infty}^{y_0} f_Y(y)\,dy = 0 - \frac{d}{dy_0}\int_{-\infty}^{T^{-1}(y_0)} f_X(x)\,dx.$$

$$f_Y(y_0) = -f_X(T^{-1}(y_0)).\frac{dT^{-1}(y_0)}{dy_0}$$

$$f_Y(y) = -f_X(T^{-1}(y))\frac{dT^{-1}(y)}{dy}$$

$$f_Y(y) = f_X\{T^{-1}(y)\}\left[-\frac{dT^{-1}(y)}{dy}\right].$$

(or)

$$\boxed{f_Y(y) = f_X(x)\left(-\frac{dx}{dy}\right)}$$

For monotonic transformation either increasing or decreasing the density function of Y is $f_Y(y) = f_X(T^{-1}(y))\left|\frac{dT^{-1}(y)}{dy}\right|$ (or)

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

Problem :- Let "X" be a continuous random variable with pdf

$$f_X(x) = \begin{cases} x/12 & 1<x<5 \\ 0 & \text{elsewhere} \end{cases}$$

find the prob density function of $Y = 2X-3$.

Sol :- Given that the pdf for the random variable "X" is

$$f_X(x) = \begin{cases} \frac{x}{12} & 1<x<5 \\ 0 & \text{elsewhere} \end{cases}$$

and also given another random variable $Y = 2X-3$

$$2X = Y+3 \Rightarrow X = \frac{Y+3}{2} \text{ (or) } x = \frac{y+3}{2}$$

Now $f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y+3}{2}\right) = \frac{1}{2}.$$

$$f_y(y) = f_x\left(\frac{y+3}{2}\right)\cdot\frac{1}{2} = \frac{\frac{1}{2}\cdot\left(\frac{y+3}{2}\right)}{12} = \frac{1}{48}(y+3).$$

If $x=1 \Rightarrow y = 2-3 = -1$

If $x=5 \Rightarrow y = 10-3 = 7$

$$f_y(y) = \begin{cases} \frac{y+3}{48} & \text{for } -1 < y < 7 \\ 0 & \text{else} \end{cases}$$

<u>Non-Monotonic</u> <u>transformation</u>:--

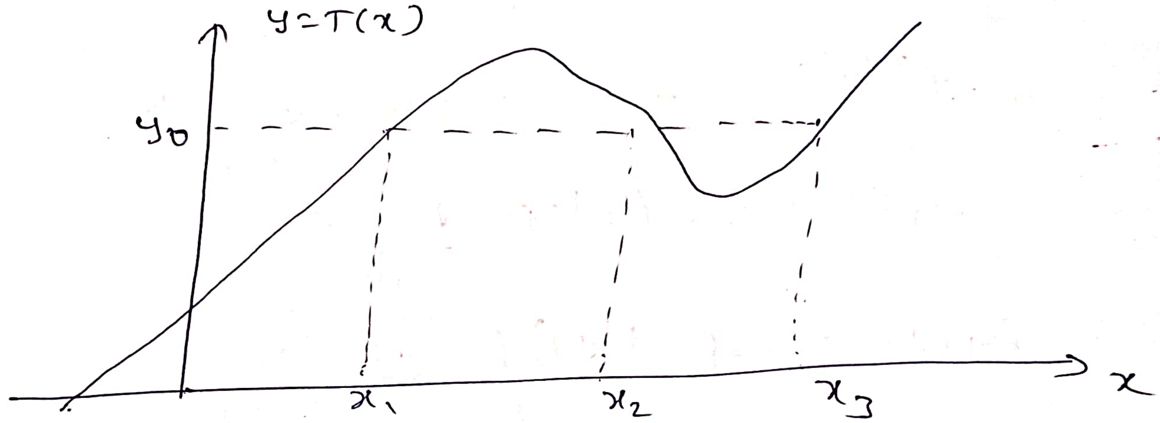


Fig: Non-Monotonic transformation.

From fig we observe that there is more than one interval of values of "x" that corresponds to the event $\{y \le y_0\}$.

→ From figure the event $\{y \le y_0\}$ corresponds to the event $\{x \le x_1$ and $x_2 \le x \le x_3\}$.

$\Rightarrow$ $P\{y \le y_0\} = P\{x \le x_1\} + P\{x_2 \le x \le x_3\}$

i.e. probability of event $\{y \le y_0\}$ is equals to the probability of the event {x values yields to $y \le y_0$}. i.e $\{x / y \le y_0\}$.

$\therefore P\{y \le y_0\} = P\{x / y \le y_0\}$

$$F_y(y_0) = \int_{x/y \le y_0} f_x(x)\,dx$$

$$\Rightarrow f_y(y_0) = \frac{d}{dy_0}\int_{x/y \le y_0} f_x(x)\,dx$$

This expression can also given as

$$f_Y(y_0) = \sum_m \frac{f_X(x_m)}{\left|\frac{d}{dx}(T(x))\right|_{x=x_m}}$$

$$\Rightarrow f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \text{----}$$

* show that the linear transformation of a gaussian r.v produces another new gaussian random variable with $y = ax+b$.

sol:- Given that $y = ax+b$

let "x" be a gaussian R.V then $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}}$

Given $y = ax+b \Rightarrow x = \frac{y-b}{a}$

we know that $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$\frac{dx}{dy} = \frac{1}{a} \Rightarrow f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{\left(\frac{y-b}{a}-a_x\right)^2}{2\sigma_x^2}}$$

$$= \frac{1}{a\sqrt{2\pi\sigma_x^2}} e^{-\frac{(y-b-a\cdot a_x)^2}{2a^2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi(a\sigma_x)^2}} e^{-\frac{(y-(b+aa_x))^2}{2(a\sigma_x)^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-a_y)^2}{2\sigma_y^2}}$$

This is also Gaussian R.V with mean $a_y = b + aa_x$ and variance $\sigma_y^2 = a^2\sigma_x^2$.

→ chebychev's inequality :-

For a given random variable "x" with mean value $\bar{x}$ and variance σ_x^2, it states that $P\{|x-\bar{x}| \geq \varepsilon\} \leq \sigma_x^2/\varepsilon^2$

where ε is very small +ve number.

Proof:- W.K.T the prob density function of a R.V x is given by

$$P\{X \le x\} = F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

Now expand $P\{|X-\bar{X}| \ge \varepsilon\} = P\{-(X-\bar{X}) \le -\varepsilon\} + P\{(X-\bar{X}) \ge \varepsilon\}$

$$= P\{-X \le \bar{X}-\varepsilon\} + P\{X \ge \bar{X}+\varepsilon\}$$

$$= \int_{-\infty}^{\bar{X}-\varepsilon} f_X(x)\,dx + \int_{\bar{X}+\varepsilon}^{\infty} f_X(x)\,dx$$

$$\therefore P\{|X-\bar{X}| \ge \varepsilon\} = \int_{|X-\bar{X}| \ge \varepsilon} f_X(x)\,dx \quad -(1)$$

Also we know that

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{X})^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{|X-\bar{X}| \ge \varepsilon} (X-\bar{X})^2 f_X(x)\,dx + \int_{|X-\bar{X}| \ge \varepsilon}^{\infty} (X-\bar{X})^2 f_X(x)\,dx$$

$$\sigma_x^2 \ge \int_{|X-\bar{X}| \ge \varepsilon}^{\infty} (X-\bar{X})^2 f_X(x)\,dx$$

if $X-\bar{X} = \varepsilon$

Then $\sigma_x^2 \ge \int_{|X-\bar{X}| \ge \varepsilon}^{\infty} \varepsilon^2 f_X(x)\,dx \Rightarrow \sigma_x^2 \ge \varepsilon^2 \int_{|X-\bar{X}| \ge \varepsilon} f_X(x)\,dx$

From eq (1)

$$\Rightarrow \sigma_X^2 \ge \varepsilon^2 P\{|X-\bar{X}| \ge \varepsilon\}$$

$$\therefore P\{|X-\bar{X}| \ge \varepsilon\} \le \frac{\sigma_X^2}{\varepsilon^2}.$$

Problem: Find the largest prob that any random variables value be smaller than its mean by 4 standard deviation or larger than its mean by the same amount.

Sol Let $\bar{X}$ be any R.V, The prob of X smaller than $\bar{X} - 4\sigma_x$ or
" " " larger " $\bar{X} - 4\sigma_x$ is given by

~~where~~ $P\{X \ge \bar{X}+4\sigma_x\} + P\{X \le \bar{X}-4\sigma_x\} = P\{|X-\bar{X}| \ge 4\sigma_x\}$

$\bar{X}$ - mean σ_x - standard deviation of x. Now using chebechev

$P\{|X-\bar{X}| \ge \varepsilon\} \le \frac{\sigma_x^2}{\varepsilon^2}$ here $\varepsilon = 4\sigma_x$ $\therefore P\{|X-\bar{X}| \ge 4\sigma_x\} = 4\sigma_x$ =

Th ... [illegible]