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(4)

UNIT - VI

FIR DIGITAL FILTERS

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INTRODUCTION: In Many Digital Processing applications FIR filters are Preferred over Their IIR Counter-Parts. The Main Advantages of FIR filter over IIR are

1. FIR filters are always stable
2. FIR filters with exactly linear phase can easily be designed
3. FIR filters can be realised in both recursive and Non-recursive structures.
4. Excellent design Methods are available for various kinds of FIR filters.

with These advantages; we have disadvantages also. The main disadvantages with FIR filters

1. Memory requirement and Execution time are very high
2. The implementation of Narrow Transmission band FIR filters are very costly,

Characteristics of FIR digital filters:-

linear phase Response of FIR filters:

The Transfer function of a FIR causal filter is given by $H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}$ —— (1)

where $h(n)$ is impulse response of filter.

let $z = e^{j\omega}$ $\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$ —— (2a)

it is a Periodic in frequency with period 2π

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j\phi(\omega)}$$ —— (2b)

where $|H(e^{j\omega})|$ = magnitude response

$\phi(\omega)$ = phase Response.

The phase delay and group delay of FIR filter is given by $\tau_p = \frac{-\phi(\omega)}{\omega}$ (3), $\tau_g = -\frac{d\phi(\omega)}{d\omega}$ (4)

for linear phase FIR filter $\phi(\omega)$ is given by

$$\phi(\omega) = -\alpha\omega \quad \text{for } -\pi \le \omega \le \pi \quad \text{— (5)}$$

where 'α' - constant phase delay in samples.

from eq (3) & (4) using Eq (5)

$$\tau_p = -\frac{(-\alpha\omega)}{\omega} = \alpha \quad , \quad \tau_g = -\frac{d(-\alpha\omega)}{d\omega} = \alpha$$

$\therefore$ $\boxed{\tau_p = \tau_g = \alpha}$ (i.e α is independent of frequency)

so from Eq (2a) & (2b)

$$\sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n} = \pm |H(e^{j\omega})| \, e^{j\phi(\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm |H(e^{j\omega})| \{\cos(\phi\omega) + j\sin\phi\omega\}$$

Comparing both real and imaginary parts.

$$\sum_{n=0}^{N-1} h(n) \cdot \cos\omega n = \pm |H(e^{j\omega})| \cdot \cos\phi(\omega) \quad \text{— (6)}$$

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \cdot \sin\phi(\omega) \quad \text{— (7)}$$

$$\frac{(7)}{(6)} \Rightarrow \frac{\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin\alpha\omega}{\cos\alpha\omega} \quad (\because \phi(\omega) = -\alpha\omega)$$

$$\Rightarrow \sin(\alpha\omega) \cdot \sum_{n=0}^{N-1} h(n) \cos(\omega n) = \cos(\alpha\omega) \cdot \sum_{n=0}^{N-1} h(n) \cdot \sin\omega n$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \left[\sin\alpha\omega \cdot \cos\omega n - \cos\alpha\omega \cdot \sin\omega n \right] = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \cdot \sin[(\alpha - n)\omega] = 0 \quad \text{— (8)}$$

$\therefore$ the Eq (8) = 0 when $h(n) = h(N-1-n)$ } for $n = 0 \rightarrow N-1$

i.e solution for Eq (8) is $\alpha = \frac{N-1}{2}$

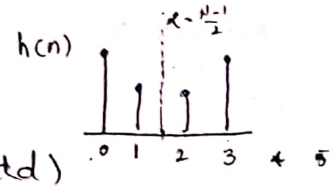
The impulse Response of FIR filter is Symmetric and have Constant phase delay and group delay. It is Symmetrical about $\alpha = \frac{N-1}{2}$.

for Even and Odd values of N the impulse-response of symmetric sequences are given as.

Let $N = 4$. (Even)

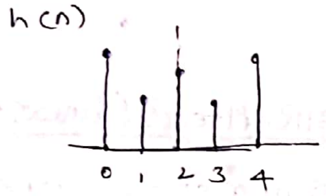
The centre of symmetry is $\alpha = \frac{N-1}{2} = \frac{3}{2}$

ex:



Let $N = 5$ (odd)

The centre of symmetry is $\alpha = \frac{N-1}{2} = 2$



* If only constant group delay is required, then

$$\phi(\omega) = \beta - \alpha\omega \quad \text{——— (9)}$$

$$\therefore \quad H(e^{j\omega}) = \pm |H(e^{j\omega})| . e^{j(\beta-\alpha\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta-\alpha\omega)}$$

$$= \pm |H(e^{j\omega})| \{ \cos(\beta-\alpha\omega) + j\sin(\beta-\alpha\omega) \}$$

$$\Rightarrow \sum_{n=0}^{N-1} \left[(\cos\omega n - j\sin\omega n) h(n) \right] = \pm |H(e^{j\omega})| \{ \cos(\beta-\alpha\omega) + j\sin(\beta-\alpha\omega) \}$$

Equating Real and Imaginary parts:

$$\sum_{n=0}^{N-1} h(n)\cos\omega n = \pm |H(e^{j\omega})| \cos(\beta-\alpha\omega) \quad \text{——— (10a)}$$

$$-\sum_{n=0}^{N-1} h(n)\sin\omega n = \pm |H(e^{j\omega})| \sin(\beta-\alpha\omega) \quad \text{——— (10b)}$$

$$\Rightarrow \frac{10b}{10a} \Rightarrow \frac{-\sum_{n=0}^{N-1} h(n)\sin\omega n}{\sum_{n=0}^{N-1} h(n)\cos\omega n} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \sin[\beta - (\alpha - n)\omega] = 0 \quad \text{——— (11)}$$

Let $\beta = \pi/2 \Rightarrow \sum_{n=0}^{N-1} h(n) \cos(\alpha-n)\omega = 0$ ——— (12)

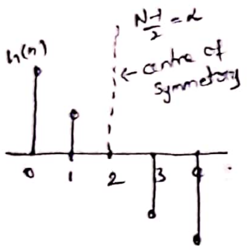
it will be satisfied when $h(n) = -h(N-1-n)$

and $\alpha = \frac{N-1}{2}$

Therefore FIR filters have constant group delay τ_g and not constant phase delay when the impulse response is Antisymmetrical about $\alpha = \frac{N-1}{2}$

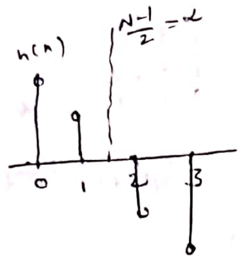
N - odd

Ex: N = 5



N - Even

Ex: N = 4



Frequency Response of FIR filters (linear phase)

Impulse Response is Symmetric and N is Odd:-

we know $z[h(n)] = H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$

let $z = e^{j\omega} \Rightarrow H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}$

let N = 5 $\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{4} h(n) e^{-j\omega n}$

$$= \sum_{n=0}^{1} h(n) e^{-j\omega n} + h(2) e^{-2j\omega} + \sum_{n=3}^{4} h(n) e^{-j\omega n}$$

$[\because \alpha = \frac{N-1}{2} = \frac{4}{2} = 2$ Centre of Symm$]$

for N samples above Eq can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{(\frac{N-3}{2})} h(n) e^{-j\omega n} + h(\tfrac{N-1}{2}) \cdot e^{-j\omega(\frac{N-1}{2})} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) \cdot e^{-j\omega n}$$

let $n = (N-1) - m$

if $n = N-1 \Rightarrow m = 0$

$n = \frac{N+1}{2} \Rightarrow m = \frac{N-3}{2}$ for symmetric

$$H(e^{j\omega}) = \sum_{n=0}^{(\frac{N-3}{2})} h(n) e^{-j\omega n} + h(\tfrac{N-1}{2}) e^{-j\omega(\frac{N-1}{2})} + \sum_{m=0}^{(\frac{N-3}{2})} h(N-1-m) \cdot e^{-j\omega(N-}$$

Replace m by n then

$$H(e^{j\omega}) = \sum_{n=0}^{(\frac{N-3}{2})} h(n) e^{-j\omega n} + h(\tfrac{N-1}{2}) e^{-j\omega(\frac{N-1}{2})} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(}$$

for impulse Response is symmetric ie $h(n) = h(N-1-n)$

0 1 2 3 4

$$\therefore H(e^{j\omega}) = \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n) \cdot e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} + h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)} h(n) \cdot e^{-j\omega(N-1-n)} e^{j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{+j\omega\left[\frac{N-1}{2} - n\right]} + h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega\left[\frac{N-1}{2} - n\right]} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cdot \cos\left[\left(\frac{N-1}{2} - n\right)\omega\right] + h\left(\frac{N-1}{2}\right) \right]$$

let $\frac{N-1}{2} - n = l$

$$\therefore H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{l=1}^{\left(\frac{N-1}{2}\right)} 2h\left[\frac{N-1}{2} - l\right] \cdot \cos[l\omega] + h\left(\frac{N-1}{2}\right) \right]$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos\omega n}$$ [by Replacing l by n and limit from $0 \to \frac{N-1}{2}$]

where $a(0) = h\left(\frac{N-1}{2}\right)$. $a(n) = 2h\left[\frac{N-1}{2} - n\right]$

$$\therefore H(e^{j\omega}) = H_1(e^{j\omega}) \cdot e^{j\phi(\omega)}.$$

where $H_1(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cdot \cos\omega n$. is real and Even function of ω

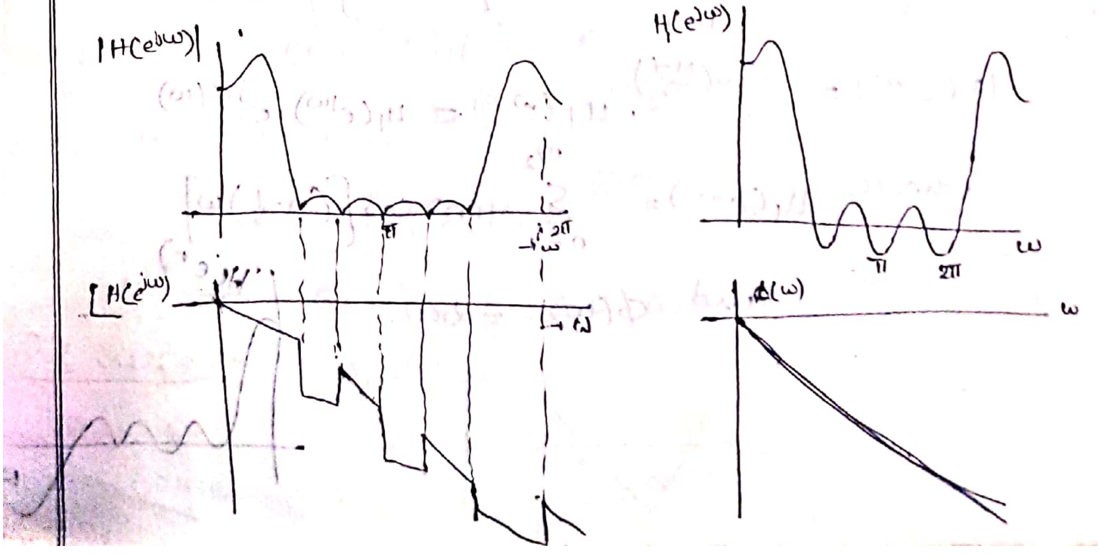
$\phi(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\alpha\omega$

$\therefore \quad |H(e^{j\omega})| = |H_1(e^{j\omega})|$

and $\angle H(e^{j\omega}) = \phi(\omega) = -\alpha\omega$ for $H_1(e^{j\omega}) \geq 0$

$= -\alpha\omega + \pi$ for $H_1(e^{j\omega}) < 0$

The Relation between magnitude and zerophase is as shown in following.



Symmetric impulse Response for N Even.

Let $N=4$

$$H(e^{j\omega}) = \sum_{n=0}^{3} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{1} h(n)\, e^{-j\omega n} + \sum_{n=2}^{3} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{4-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{4-1} h(n)\, e^{-j\omega n}$$

In General we can write

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

Let $N-1-m=n$ then

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

Replacing m by n, we get

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right] \quad \left[\because h(n) = h(N-1-n)\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\frac{N}{2}-1} h(n) \right\} \left\{ e^{+j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cdot \cos\left[\left(\frac{N-1}{2}-n\right)\omega\right] \right]$$

Let $\frac{N-1}{2} - n = M$ and Replace it by we got

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \cdot \cos\left[\left(n-\frac{1}{2}\right)\omega\right] \right]$$

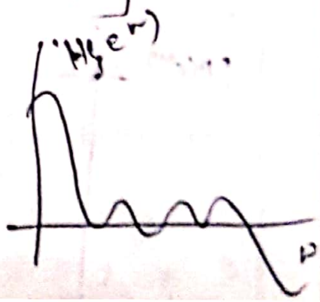
$$= \left(e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=1}^{\frac{N}{2}} b(n) \cdot \cos\left[\left(n-\frac{1}{2}\right)\omega\right] \right), \quad b(n) = 2h\left(\frac{N}{2}-\right.$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot H_1(e^{j\omega}) = H_1(e^{j\omega}) \cdot e^{j\phi(\omega)}$$

where

$$H_1(e^{j\omega}) = \sum_{n=1}^{N/2} y(n) \cos\left[\left(n-\frac{1}{2}\right)\omega\right]$$

and $\phi(\omega) = -\alpha\omega$.



④

Impulse response is Antisymmetric and N-odd

as similar to symmetric we may find Antisymmetric for both N-Even and odd,

for This case $h\left(\frac{N-1}{2}\right)=0$

we know $H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \cancel{h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)}} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

But $h(n) = -h(N-1-n)$ Then

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} \left\{ h(n)\, e^{-j\omega n} - h(n)\cdot e^{-j\omega(N-1-n)} \right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left\{ e^{-j\omega n} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} - e^{-j\omega(N-1-n)} \cdot e^{j\omega\left(\frac{N-1}{2}\right)} \right\} \right]$$

after simplification we get $2j\sin\theta$, $j = e^{j\pi/2}$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\frac{\pi}{2}} \sum_{n=1}^{\frac{N-1}{2}} 2h(n)\cdot \sin\omega\left(\frac{N-1}{2}-n\right)$$

$$= e^{j\frac{\pi}{2}} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cdot \sin n\omega$$

$$\therefore \boxed{H(e^{j\omega}) = e^{j\frac{\pi}{2}} \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin\omega n}$$

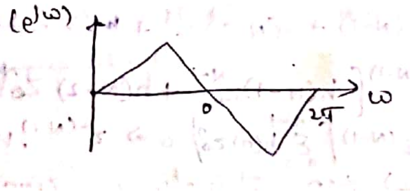
$$\boxed{\text{where } c(n) = 2h\left(\frac{N-1}{2}-n\right)}$$

$$H(e^{j\omega}) = H_1(e^{j\omega}) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\frac{\pi}{2}}$$

$$= H_1(e^{j\omega})\cdot e^{j\phi(\omega)}$$

where $\phi(\omega) = \frac{\pi}{2} - \alpha\omega$ and $H_1(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin\omega n$.

Fr response



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Impulse Response is Assymmetric and N-Even:

Similarly we get

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot e^{j\frac{\pi}{2}} \left[\sum_{n=1}^{N/2} d(n) \sin\omega\left(n-\tfrac{1}{2}\right) \right]$$

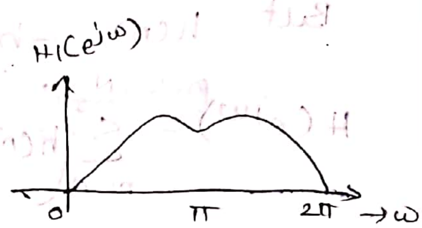
where $d(n) = 2h\left(\frac{N}{2}-n\right)$

$$H(e^{j\omega}) = e^{j\theta(\omega)} \cdot H_1(e^{j\omega})$$

where $H_1(e^{j\omega}) = \sum_{n=1}^{N/2} d(n) \sin\left(n-\tfrac{1}{2}\right)\omega$

$$\phi(\omega) = \frac{\pi}{2} - \alpha\omega$$

freq response of linear phase is



Notes:

* odd 'N' The impulse response of Symmetric with odd can be used to design all the filters. (types)

* Symmetric impulse with Even N cannot be used to design high pass filter.

* The frequency response of Anti-Symmetric is imaginary to within a linear phase factor. This case of filters are most suitable for such filter as Hilbert Transformers and differentiators.

Zeros of Linear phase FIR filter:

The T.F of linear phase FIR filter is given by $H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$

Let $z_0 \neq 0$ is finite zero of $H(z)$ then

$$H(z)\Big|_{z=z_0} = H(z_0) = \sum_{n=0}^{N-1} h(n) z_0^{-n} = 0$$

$$= h(0) + h(1) z_0^{-1} + h(2) z_0^{-2} + \cdots + h(N-1) z_0^{-(N-1)} = 0$$

for linear phase $h(n) = h(N-1-n)$

so $H(z)\Big|_{z=z_0} = h(N-1) + h(N-2) z_0^{-1} + \cdots + h(1) z_0^{-(N-2)} + h(0) z_0^{-(N-1)} = 0$

$$= z^{-(N-1)} \left[h(N-1) z_0^{N-1} + h(N-2) z_0^{N-2} + \cdots + h(1) z_0^{1} + h(0) \right] = 0$$

$$= z^{-(N-1)} \left[\sum_{n=0}^{N-1} h(n) z_0^{n} \right] = 0 \Rightarrow z^{-(N-1)} H(z_0^{-1}) z_0 \Rightarrow \underline{H(z) = H(z_0^{-1})} =$$

if z_0 is zero of $H(z)$ then z_0^{-1} is also a zero.

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REALIZATION OF FIR DIGITAL FILTERS

The basic Realization structures of FIR filters are
1. Direct form Realization
2. Cascade form Realization
3. Linear phase Realization.

Direct form Realization:

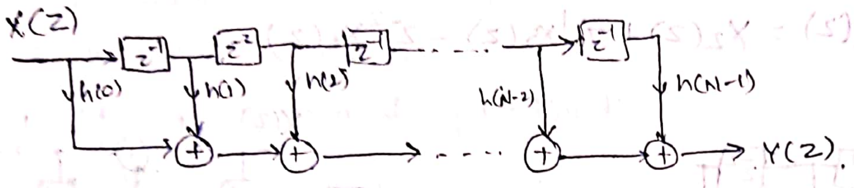
The System function of an FIR filter is

$$H(z) = \sum_{n=0}^{N-1} h(n)\cdot z^{-n}$$

$$\frac{Y(z)}{X(z)} = h(0) + h(1)z^{-1} + h(2)z^{-2} + \dots + h(N-1)z^{-(N-1)}$$

$$\Rightarrow Y(z) = h(0)\cdot X(z) + h(1)z^{-1}X(z) + h(2)z^{-2}X(z) + \dots + h(N-1)z^{-(N-1)}X(z) \quad \text{--- (1)}$$

Eq (1) can be realized as



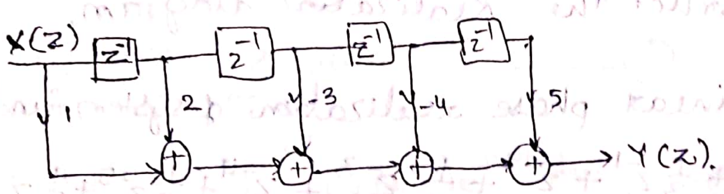
Ex: Determine Directform Realization of the System

$$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

Sol: $H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

$$\Rightarrow Y(z) = X(z) + 2z^{-1}X(z) - 3z^{-2}X(z) - 4z^{-3}X(z) + 5z^{-4}X(z)$$

The above Eq can be realized as



Ex: Determine direct form Realization of ~~structure~~ System.

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5}$$

→ Find freq Response of System (FIR) described by the difference eq $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$. and also calculate the phase delay and group delay.

CASCADE FORM REALIZATION :

If H(z) is given then that will be divided into two parts. for Equa Each part apply Direct form realization Then cascade both realization.

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{N-1} [h_1(n) * h_2(n)] z^{-n}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

Ex: Obtain cascade form realization of system

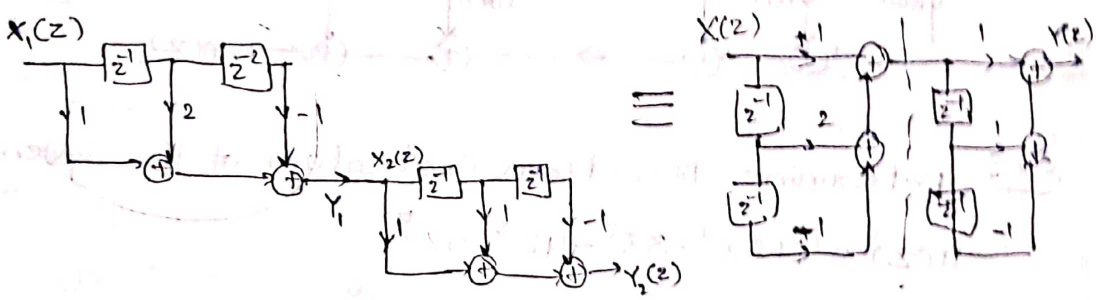
$$H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$$

Let $H(z) = H_1(z) \cdot H_2(z)$

$\Rightarrow H_1(z) = 1+2z^{-1}-z^{-2}$, $H_2(z) = 1+z^{-1}-z^{-2}$

$\Rightarrow Y_1(z) = X_1(z) + 2z^{-1}X_1(z) + z^{-2}X_1(z)$

$\Rightarrow Y_2(z) = X_2(z) + z^{-1}X_2(z) - z^{-2}X_2(z)$



LINEAR PHASE REALIZATION :

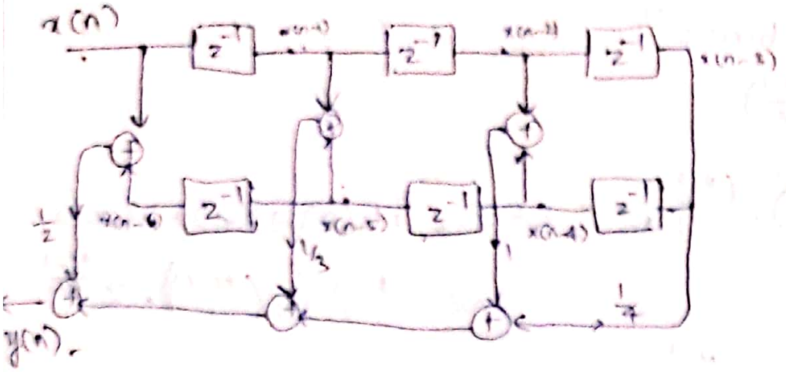
find the Realization of given system function and consider the present values as symmetric Then construct the Realization diagram.

Ex: obtain linear phase realization of system function

$$H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$$

$\Rightarrow Y(z) = \frac{1}{2}X(z) + \frac{1}{3}z^{-1}X(z) + z^{-2}X(z) + \frac{1}{4}z^{-3}X(z) + z^{-4}X(z) + \frac{1}{3}z^{-5}X(z) + \frac{1}{2}z^{-6}X(z)$

$\Rightarrow y(n) = \frac{1}{2}x(n) + \frac{1}{3}x(n-1) + x(n-2) + \frac{1}{4}x(n-3) + x(n-4) + \frac{1}{3}x(n-5) + \frac{1}{2}x(n-6)$



Ex: obtain linear phase Realization of system $H(z)$

$$H(z) = 1 + \frac{1}{2}z^{-1} + 3z^{-2} + 3z^{-3} + \frac{1}{2}z^{-4} + z^{-5}$$

DESIGNING OF FIR FILTERS:-

FIR filters canbe designed by using 3 Methods.

1. Fourier Series Method.
2. Window Technique.
3. Frequency Sampling Method.

Fourier Series Method:

$H(e^{j\omega})$ [Frequency Response] of a system is periodic with in 2π. so from Fourier series analysis we know that any periodic function can be Expressed as a linear Combination of Complex Exponentials. So The Frequency response of an FIR filter can be represented by Fourier Series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-j\omega n} \quad \text{——— (1)}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega \quad \text{——— (2)}$$

z-transform of a given sequence is : $H(z) = \sum_{n=-\infty}^{\infty} h_d(n) \cdot z^{-n}$ ——— (3)

Eq (3) represents a Non-causal digital filter of infinite duration. To get an FIR filter T.F, Series can be Truncated by

$$\left.\begin{aligned} h(n) &= h_d(n) \quad \text{for } |n| \le \frac{N-1}{2} \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \quad \text{——— (4)}$$

Then

$$H(z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) z^{-n}$$

$$= h\left(\frac{N-1}{2}\right) z^{-\left(\frac{N-1}{2}\right)} + \dots + h(1) z^{-1} + h(0) + h(-1) + h(-2) + \dots + h\left(-\left(\frac{N-1}{2}\right)\right) \cdot z^{\frac{N-1}{2}}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) z^{-n} + h(-n) z^{+n}\right]$$

for Symmetric impulse response at $n=0$ is $h(n) = h(-n)$

$$\therefore H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{-n} + z^{n}\right] \quad \text{---} \; (5)$$

To get Realizable form. Eq multiply above Eq with $z^{-\left(\frac{N-1}{2}\right)}$ where $\frac{N-1}{2}$ is delay in samples.

$$\therefore H'(z) = z^{-\left(\frac{N-1}{2}\right)} \cdot H(z)$$

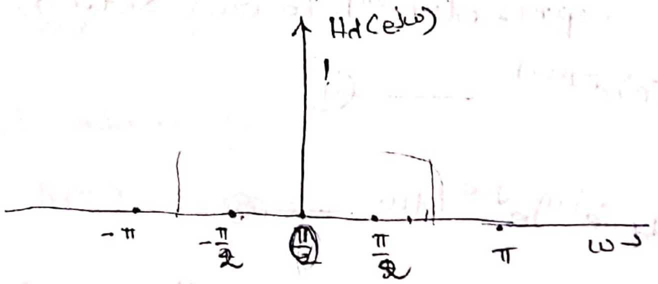
$$= z^{-\left(\frac{N-1}{2}\right)} \left\{ h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{-n} + z^{n}\right] \right\}.$$

Ex: Design an Ideal LPF with a frequency Response of

$$H_d(e^{j\omega}) = 1 \text{ for } -\frac{\pi}{2} \le |\omega| \le \frac{\pi}{2}$$
$$= 0 \text{ for } \frac{\pi}{2} \le |\omega| \le \pi.$$

find values of $h(n)$ for $N=11$. find $H(z)$

Sol: freq response of LPF with $\omega_c = \frac{\pi}{2}$ is



from freq response $\alpha = 0$ so we get Non-causal filter Coefficients Symmetric about $n=0$.

i.e. $h_d(n) = h_d(-n)$

The filter coefficients can be obtained for zero phase response as follows

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$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\pi/2} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi(jn)}\left[e^{j\omega n}\right]_{-\pi/2}^{\pi/2} = \frac{1}{n\pi(2j)}\left[e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right]$$

$$\Rightarrow h_d(n) = \frac{\sin\frac{\pi}{2}n}{n\pi} \qquad -\infty \le n \le \infty$$

Truncating $h_d(n)$ for $N=11$ Samples.

$$h(n) = \frac{\sin\frac{\pi}{2}n}{\frac{2n\pi}{2}} \quad \text{for } |n| \le 5$$

$$= 0 \quad \text{otherwise}$$

$\underline{n=0}$

$$h(0) = \frac{\sin\frac{\pi}{2}n}{n\pi} = \lim_{n\to 0} \frac{1}{2}\cdot\frac{\sin\frac{\pi}{2}n}{\frac{n\pi}{2}} \quad \left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right)$$

$$= \frac{1}{2}$$

$$h(1) = h(-1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.3183$$

$$h(2) = h(-2) = \frac{\sin\pi}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin 3\pi/2}{3\pi/2} = -\frac{1}{3\pi} = -0.106$$

$$h(4) = h(-4) = \frac{\sin 4\pi/2}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin 5\pi/2}{5\pi} = \frac{1}{5\pi} = 0.06366$$

So T.F of filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{-n} + z^{n}\right].$$

$$= 0.5 + 0.3183(z^{1} + z^{-1}) - 0.106(z^{3} + z^{-3}) + 0.06366(z^{5} + z^{-5})$$

The T.F of Realizable filter is $H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z)$

$$H'(z) = z^{-5}\left[H(z)\right]$$

$$= 0.06366 - 0.106 z^{-2} + 0.318 z^{-4} + 0.5 z^{-5} + 0.3183 z^{-6} - 0.106 z^{-8} + 0.0636 z^{-10}$$

$\therefore$ filter Coefficients given by

$h(0) = h(10) = 0.0636$

$h(1) = h(9) = 0$

$h(2) = h(8) = -0.106$

$h(3) = h(7) = 0$

$h(4) = h(6) = 0.3183$

$h(5) = 0.5$

$\therefore$ frequency response is given by

$$H_1(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a_n \cos \omega n \quad \text{where} \quad a(0) = h\left(\frac{N-1}{2}\right)$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

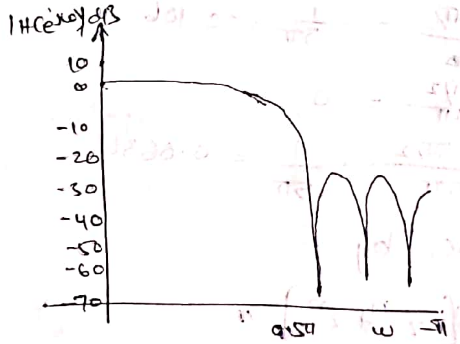
$\therefore$ $a(0) = 0.5$, $a(1) = 0.6366$, $a(4) = 0$

$a(2) = 0$

$a(3) = -0.212$, $a(5) = 0.127$

$\therefore$ $H_1(e^{j\omega}) = 0.5 + 0.6366 \cos\omega - 0.212 \cos 3\omega + 0.127 \cos 5\omega$

magnitude $|H(e^{j\omega})|_{dB} = 20 \log |H_1(e^{j\omega})|$ where $0 \le \omega \le \pi$



freq response of LPF

Similarly remaining filter designs also.

* filters are of 2 types depending on Impulse Response. They are 1. IIR filters 2. FIR filters.
IIR filters are of Type-I filters (Recursive)
FIR filters are of Non-Recursive filters.
* filters are of 4 types based on frequency Response.
1. LPF 2. HPF 3. BPF 4. BRF.
* For a linear phase filter $\phi(\omega) \propto \omega$, linear phase filter doesnot alter the shape of the original signal. If the phase Response of the filter is Non-linear The output signal may be distorted one.
In Many cases a linear phase characteristics is Required Throughout the passband of the filter to Preserve ( ) The shape of a given signal within the passband. An IIR filter can't produce a linear phase the FIR filter can give linear phase, when impulse Respons of filter is Symmetric about its mid point.

* Desirable & undesirable features of FIR filter
1) FIR filters have exact linear phase
2) FIR filters always stable
3) FIR filter can be realized in both Recursive & Non-Recursive structures.

(a)
1) for same ~~for~~ filter Specifications The order of FIR filter design can be as high as 5 → 10 times IIR.
2. large storage requirement needed
3. powerful computational facilities Required for The implementation

Filter Coefficients of FIR filters

Type	Coefficients of linear phase filter with delay $\alpha = \frac{N-1}{2}$
1. LPF with cutoff frequency ω_c	$h_d(n) = \frac{\omega_c}{\pi}$ for $n = \alpha$. $= \frac{\sin \omega_c(n-\alpha)}{\pi(n-\alpha)}$ for $n \neq \alpha$
2. HPF with cutoff frequency ω_c	$h_d(n) = 1 - \frac{\omega_c}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c]$ $n \neq \alpha$
3. BPF with cutoff frequencies ω_{c_1} and ω_{c_2}	$h_d(n) = \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin \omega_{c_2}(n-\alpha) - \sin \omega_{c_1}(n-\alpha)]$
4. BRF with cutoff frequencies ω_{c_1} and ω_{c_2}	$h_d(n) = 1 - \frac{\omega_{c_2} - \omega_{c_1}}{\pi}$ for $n = \alpha$ $= \frac{1}{\pi(n-\alpha)}[\sin \omega_{c_1}(n-\alpha) - \sin \omega_{c_2}(n-\alpha) + \sin(n-\alpha)\pi]$

DESIGN OF FIR FILTERS USING WINDOWS:

We have various window functions.

1. Rectangular window
2. Hamming window
3. Hanning window
4. Blackman window
5. Bartlett window
6. Kaiser window

Rectangular window :-

The Rectangular window sequence is given as

$W_{Rec}(n) = 1$, for $n = 0$ to $N-1$ (or)

$n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$

$= 0$, elsewhere.

The Rectangular window spectrum is given as.

$$W_{Rec}(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} e^{-j\omega n} = e^{j\omega\left(\frac{N-1}{2}\right)} + \dots + 1 + e^{-j\omega} + e^{-2j\omega} + \dots + e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)}\left[1 + e^{-j\omega} + \dots + e^{-j\omega(N-1)}\right]$$

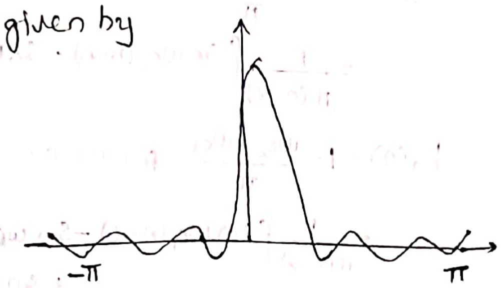
$$\therefore\ 1+e^{-j\omega}+e^{-2j\omega}+\dots+e^{-j\omega(N-1)} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$W_{Rec}(e^{j\omega}) = e^{j\omega\left(\frac{N-1}{2}\right)}\left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right]$$

$$= \frac{e^{j\frac{\omega N}{2}}\left[1-e^{-j\omega N}\right]}{e^{j\frac{\omega}{2}}(1-e^{-j\omega})} = \frac{\left(e^{j\frac{\omega N}{2}}-e^{-j\frac{\omega N}{2}}\right)/2j}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}/2j}$$

$$= \frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$, where N - length of Sequence

The frequency response of N-pt rectangular window is given by



Ex: Design a LPF using Rectangular window by taking 9 Samples of w(n) and with a cut-off frequency of 1.2 radians/sec.

Sol:- given $\omega_c = 1.2$ radians/sec. (cut-off frequency)

Number of samples, $N = 9$.

The desired frequency response of LPF is given by

$$H_d(\omega) = e^{-j\omega\alpha}, \quad -\omega_c \le \omega \le \omega_c$$

$$= 0 \quad \text{elsewhere.}$$

$$FT[h_d(n)] = H_d(\omega)$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha}\cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\alpha)}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi[j(n-\alpha)]}\left[e^{j(\omega_c(n-\alpha))} - e^{-j(\omega_c(n-\alpha))}\right]$$

$$= \frac{1}{\pi[(n-\alpha)]}\left[\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right]$$

$$= \frac{1}{\pi[(n-\alpha)]} \sin(\omega_c(n-\alpha))$$

if $n=\alpha$, $\frac{\sin(\omega_c(n-\alpha))}{n-\alpha} = \frac{0}{0}$ is an indeterminate form.

$hd(n) = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}$, for $n \neq \alpha$

Applying L'Hospital rule, the value of $hd(n)$ is Evaluated for $n=\alpha$

$\therefore$ for $n=\alpha$, $hd(n) = \lim_{n\to\alpha} \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)} = \frac{\omega_c}{\pi}$

The impulse Response of filter is $\boxed{h(n) = hd(n) \cdot W_{Rec}(n)}$

$h(n) = hd(n)$ for $n = 0$ to $N-1$

given $N=9$, $\omega_c = 1.2$ rad/sec

$\alpha = \frac{N-1}{2} = 4$

$n=0 \Rightarrow h(0) = \frac{\sin(1.2(0-4))}{\pi(0-4)} = -0.0793$

$n=1 \Rightarrow h(1) = \quad = -0.047$

$n=2 \Rightarrow h(2) = \quad = 0.1075$

$n=3 \Rightarrow h(3) = \frac{\omega_c}{\pi} \quad = 0.297$

$n=4 \Rightarrow h(4) = \quad = 0.382$

$n=5 \Rightarrow h(5) = \quad = 0.297$

$n=6 \Rightarrow h(6) = \quad = 0.1075$

$n=7 \Rightarrow h(7) = \quad = -0.047$

$n=8 \Rightarrow h(8) = \quad = -0.0793$

2, 4, 7, 11, 14, 19, 26, 31,
32, 38, 39, 43, 47, 48, 49
50, 57, 1, 1, 1

from above values

$h(0)=h(8)$, $h(1)=h(7)$, $h(2)=h(6)$, $h(3)=h(5)$

$\therefore$ h(n) satisfying symmetric condition $h(n)=h(N-1-n)$

for N-odd symmetric impulse response, the magnitude function is

$$|H(e^{j\omega})| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos n\omega$$

$$= h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$= 0.382 + 0.594\cos\omega + 0.215\cos 2\omega - 0.094\cos 3\omega - 0.1586\cos 4\omega$$

System function is $H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$

$$\Rightarrow H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6} + h(7)z^{-7} + h(8)z^{-8}$$

- Apply symmetric condition

$$H(z) = h(0)[1+z^{-8}] + h(1)[z^{-1}+z^{-7}] + h(2)[z^{-2}+z^{-6}] + h(3)[z^{-3}+z^{-5}] + h(4)\cdot z^{-4}$$

Hanning window :

The hanning window sequence can be obtained by substituting $\alpha=0.5$ in Raised cosine window

Eq

$$W_\alpha(e^{j\omega}) = \alpha\frac{\sin\omega\frac{N}{2}}{\sin\omega/2} + \frac{1-\alpha}{2}\cdot\frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))} + \frac{(1-\alpha)}{2}\frac{\sin(\omega N/2 + \pi N/(N-1))}{\sin(\omega/2 + \pi/(N-1))}.$$

Then Hanning window

$$W_{Hn}(n) = 0.5 + 0.5\cos 2\pi n/(N-1) \quad \text{for } -(N-1)/2 \le n \le (N-1)/2$$

$$= 0 \quad \text{otherwise}$$

The freq response of Hanning window is

$$W_{Hn}(e^{j\omega}) = 0.5\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} + 0.25\frac{\sin(\omega N/2 - \pi N/(N-1))}{\sin(\omega/2 - \pi/(N-1))}$$

t·

...a causal. $W_{Rec}(n) = \begin{cases} 1 & \text{for } n=0 \text{ to } N-1 \\ 0 & \text{elsewhere} \end{cases}$

Non-causal.

$$W_{Rec}(n) = \begin{cases} 1 & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{elsewhere,} \end{cases}$$

...mming window

Causal:

$$W_{Ham}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) & \text{for } n = 0,1,2,\cdots N-1. \\ 0 & \text{elsewhere} \end{cases}$$

non-causal:

$$W_{Ham}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right) & \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right) \\ 0 & \text{elsewhere.} \end{cases}$$

...ckman windows:-

Causal.

$$W_{Bla}(n) = 0.42 - 0.5\cos\left(\frac{2n\pi}{N-1}\right) + 0.08\cos\left(\frac{4n\pi}{N-1}\right)$$

$$n = 0,1,\cdots(N-1)$$

...iser window:-

$$W_{Kai}(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{n}{\left(\frac{N-1}{2}\right)}\right)^2}\right]}{I_0\alpha} \quad \text{for } n = -\left(\frac{N-1}{2}\right) \text{ to } \left(\frac{N-1}{2}\right)$$

$= 0$ elsewhere.

Frequency Sampling Method of designing FIR filters.

let $h(n)$ is filter coefficients of an FIR filter and $H(K)$ is the DFT of $h(n)$. Then we have $h(n) = \frac{1}{N}\sum_{K=0}^{N-1} H(K) \cdot e^{\frac{j2\pi Kn}{N}}$, $n = 0,1,2 \ldots N-1$ —— (1)

and $H(K) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\frac{2\pi Kn}{N}}$, $K = 0,1,\ldots,N-1$, —— (2)

The DFT samples $H(K)$ for an FIR sequence can be Regarded as samples of filter z-Transform Evaluated at N-points Equally spaced around unit circle,

i.e. $H(K) = H(Z)\Big|_{z = e^{\frac{j2\pi K}{N}}}$ —— (3)

The Transfer function $H(Z)$ of an FIR filter with impulse Response is given by $H(Z) = \sum_{n=0}^{N-1} h(n) z^{-n}$ —— (4)

Substituting Eq (1) in Eq (4).

$$H(Z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} H(K) \cdot e^{j2\pi Kn/N}\right] \cdot z^{-n}$$

$$= \sum_{K=0}^{N-1} \frac{H(K)}{N} \cdot \sum_{n=0}^{N-1} \cancel{H(K)} \left[e^{j2\pi K/N} \cdot z^{-1}\right]^{n}$$

$$= \sum_{K=0}^{N-1} \frac{H(K)}{N} \cdot \frac{\left[1 - (e^{j2\pi K/N} \cdot z^{-1})^{N}\right]}{\left[1 - e^{j2\pi K/N} \cdot z^{-1}\right]} = \frac{1-z^{-N}}{N}\sum_{K=0}^{N-1} \frac{H(K)}{1 - e^{j2\pi K/N} \cdot z^{-1}} \quad \text{—— (5)}$$

But we know $H(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{N}} = H(e^{j2\pi Kn/N}) = H(K)$. —— (6)

i.e $H(K)$ is the K^{Th} DFT component obtained by Sampling the frequency response $H(e^{j\omega})$. Such approach of designing FIR filter is called frequency Sampling method.

Frequency Sampling Realization:

Eq (5) can be written as $H(Z) = \frac{1-z^{-N}}{N}\sum_{K=0}^{N-1} G_K(z)$

where $G_K(z) = \frac{H(K)}{1 - e^{\frac{j2\pi K}{N}} \cdot z^{-1}}$ $0 \le K \le N-1$. it is the Transfer function of 1^{st} order FIR filters.