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UNIT-V

IIR DIGITAL FILTERS

Dt: 3-3-09

INTRODUCTION: Basically a Digital filter is a linear time-invarient discrete-time system. we have Two types of filters. They are (i) IIR (ii) FIR

FIR filters are of Non-recursive type. where The present output sample depends on Present input sample and Previous input sample.

IIR filters are of Recursive-type where the Present output sample depends on the prese-nt input, past input and output Sample.

The impulse Response h(n) for a filter is. $h(n) = 0$ for $n \le 0$. and for stability it must be satisfy The Condition $\sum_{n=0}^{\infty} |h(n)| < \infty$

IIR digital filters have the following Transfer function of The form H(z)

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{\sum_{K=0}^{M} b_K z^{-K}}{1 + \sum_{K=1}^{N} a_K z^{-K}} \quad \text{———} (1)$$

for design of digital filter we must find The Coefficients in Eq(1) are b_K & a_K.

~~The~~ so before going to know about design of digital filter we must have The Minimum know-ledge about filters. (know The following definitions);

Filter: A filter is one, which rejects unwan-ted frequencies from input signal and allow desired frequencies to obtain The required shape of output signal

Passband: The Range of frequencies of signal That are passed Through filter.

Stopband: The Range of frequencies whose Signals That are blocked

The filters can be divided into following:

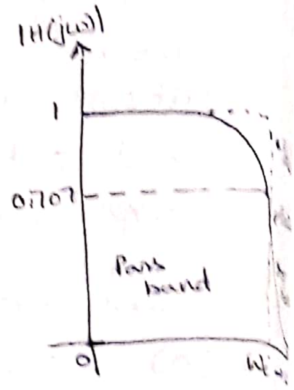
(1) Lowpass filter.

(2) Highpass filter.

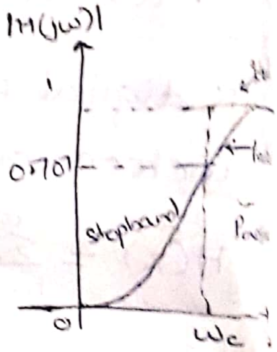
(3) Band Pass filter.

4) Band Reject filter.

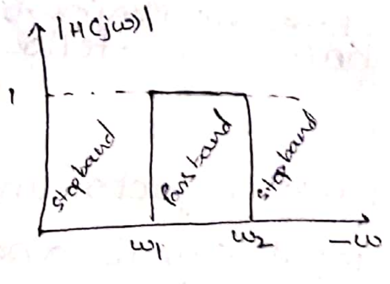
Low Pass filter: The Magnitude response of an ideal lowpass filter allows low frequencies in passband $0 < \omega < \omega_c$ to pass whereas the higher frequencies in stopband $\omega > \omega_c$ are blocked. The frequency ω_c between 2 bands is called cut-off freq.

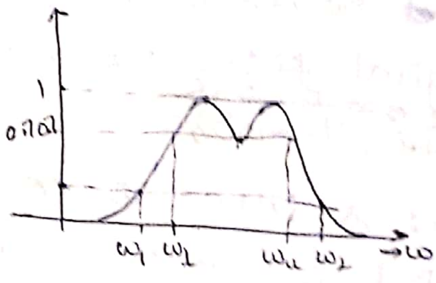


Highpass filter: The Highpass filter allows high frequencies above $\omega > \omega_c$ and Rejects the frequencies between $\omega = 0$ and $\omega = \omega_c$.

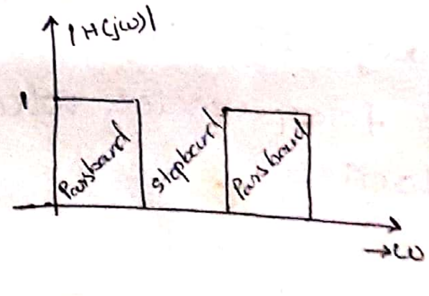


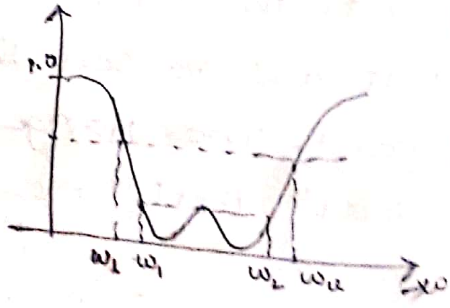
Bandpass filter: It allows only a band of frequencies ω_1 to ω_2 to pass and stop all other freq.





Band reject filter: It rejects all the freq between ω_1 and ω_2 and allows remaining frequencies





DIGITAL Vs ANALOG FILTERS:

Analog filter	Digital filter
1. Analog filter Processes analog inputs and Generates analog o/ps	1. A digital filter Processes and Generates digital data.
2. Analog filters are Constructed from Active or Passive Electronic-Components	2. It consists of Elements like adder, Multiplier and delay unit
3. It is described by a differential Equation	3. It is described by a difference Equation
4. ~~Its~~ frequency response of an analog filter Can be Modified by changing the Components	4. It's frequency Response can be changed by changing the filter coefficients.

Advantages of digital filters:

1. Digital filter performance is not influenced by Temperature and power Supply variations.
2. No problems of i/p or o/p impedance matching
3. Digital filters can operate over a wide-Range of frequencies.
4. Multiple filtering possible only in digital filter.

Only one disadvantage with digital filter i.e The quantization Error arises due to finite word length in representations of signals and Parameters.

Analog Lowpass filter design: The most General form of analog filter Transfer function is $H(S)$.

$$H(S) = \frac{N(S)}{D(S)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i} \quad \text{——— (1)}$$

In This chapter we replace $\omega \rightarrow \Omega$

where $H(s)$ is the Laplace Transform of impulse Response $h(t)$, i.e. $H(s) = \int_{-\infty}^{\infty} h(t).e^{-st}dt$. and $N \geq M$ must be satisfied.

for a stable analog filter, the poles of $H(s)$ lies in the left half of s-plane.

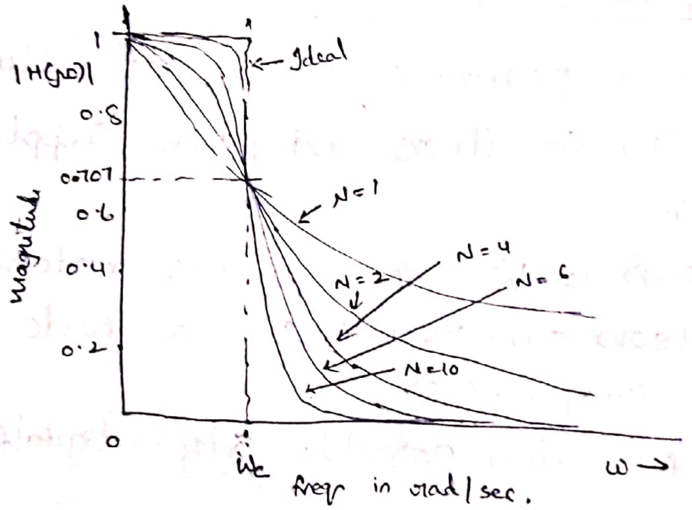
we have Two types of Analog filter designs. They are 1. Butterworth filter 2. chebyshev filter.

Analog Low Pass Butterworth filter :-

The Magnitude function of Butterworth LPF is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2N}}} \quad \text{where } N = 1,2,3\ldots \quad (1)$$

where
N - order of filter
ω_c - cutoff frequency.



In above figure we can observe that the maximum response is unity at $\omega = 0$. we can observe that the Magnitude response approaches the Ideal lowpass filter characteristics as the order 'N' ~~decreases~~ increases.

for $\omega < \omega_c$ Then $|H(j\omega)| = 1$
$\omega > \omega_c$ Then $|H(j\omega)| \rightarrow$ decreases Rapidly
$\omega = \omega_c$ Then curve pass Through 0.707, which corresponds 3dB point.

for Normalized Butterworth filter $\omega_c = 1$ rad/sec.

$\therefore$ from Eq (1)

$$|H(j\omega)|^2 = \frac{1}{1+(\omega)^{2N}} \quad , N=1,2,3\ldots \quad \text{——— (1k)}$$

So the Transfer function of stable filter, is obtained by substituting $\omega = \frac{s}{j}$

$$\therefore |H(j\omega)|^2 = |H(j\omega)|\cdot|H(-j\omega)|$$

$$= H(j\omega)\cdot H(-j\omega)$$

$$= \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

$$\Rightarrow H(s)\cdot H(-s) = \frac{1}{1+(-1)^N s^{2N}} = \frac{1}{1+(-s^2)^N} \quad \text{——— (2)}$$

from Eq (2) we can Conclude That, The function has Poles in both Left & Right half of s-plane. due to Presence of $H(s)$ & $H(-s)$., $\because H(s)$ has roots in LHP,

So, The Roots can obtained by Equating Denominator to zero. i.e. $1+(-s^2)^N = 0$ ——— (3)

Let N – odd

from (3) $\Rightarrow$ $s^{2N} = 1$

$$= e^{j2\pi K}$$

$$\Rightarrow S_K = e^{j\pi K/N} \quad , K = 1,2\ldots 2N \quad \text{——— (4)}$$

$z = e^{j\omega}$

$z = e^{s}$

$s = j\omega$.

for N-Even:

$$s^{2N} = -1$$

$$= e^{j(2K-1)\pi}$$

$$\Rightarrow S_K = e^{j(2K-1)\frac{\pi}{2N}} \quad \text{for } K = 1,2\ldots 2N \quad \text{——— (5)}$$

for $N=3$ Eq (3) becomes as $s^6 = 1$

$\therefore$ when N-odd roots obtained by Eq (4)

$$\Rightarrow S_1 = e^{j\pi/3} = \cos\pi/3 + j\sin\pi/3 = 0.5 + j0.866$$

$$\Rightarrow S_2 = e^{j2\pi/3} = \cos 2\pi/3 + j\sin 2\pi/3 = -0.5 + j0.866$$

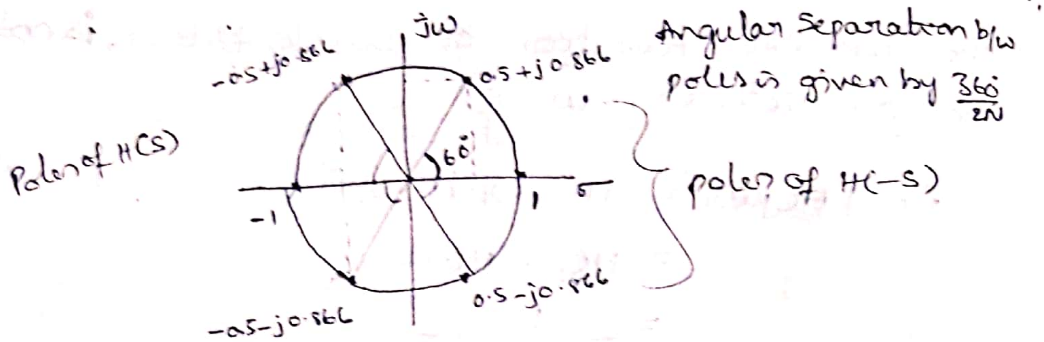
$$\Rightarrow S_3 = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$S_4 = e^{j4\pi/3} = -0.5 - j0.866$

$S_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j0.866$

$S_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1$

∴ The above roots are located in s-plane as follows.



for Stability let the poles only lies in left half of s-plane

So denominator of Transfer function H(S) as

$\Rightarrow (S+1)(S-(-0.5+j0.866))(S-(-0.5-j0.866))$

$\Rightarrow (S+1)((S+0.5)+j0.866)((S+0.5)+j0.866))$

$\Rightarrow (S+1)((S+0.5)^2+(0.866)^2)$

$\Rightarrow (S+1)(S^2+S+1)$.

∴ The Transfer function of 3rd order Butterworth filter ($\omega_c = 1$ rad/sec) is $H(S) = \frac{1}{(1+S)(1+S^2+S)}$

The same left side poles can be found by $S_k = e^{j\phi_k}$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2, 3 \ldots N$. —(6)

Let N = 4

Eq (4) becomes $S^8 = 1$

when N-Even roots obtained from Eq (5)

$S_k = e^{j(2k-1)\pi/2N}$

$S_1 = e^{j\frac{\pi}{8}} = 0.9239 + j0.3827$

$S_2 = e^{j3\pi/8} = 0.3827 + j0.9239$

✓ $S_3 = e^{j5\pi/8} = -0.3827 + j0.9239$

✓ $S_4 = e^{j7\pi/8} = -0.9239 + j0.3827$

✓ $S_5 = e^{j9\pi/8} = -0.9239 - j0.3827$

✓ $S_6 = e^{j11\pi/8} = -0.3827 - j0.9239$

$S_7 = e^{j13\pi/8} = 0.3827 - j0.9239$

$S_8 = e^{j15\pi/8} = 0.9239 - j0.3827$

for stability let left poles of s-plane.

$\therefore$ denominator of $H(S)$ is given by

$$= ((s+0.3827)-j0.9239)((S+0.9239)-j0.3827)((S+0.9239)+j0.3827) \cdot ((S+0.3827)+j0.9239)$$

$$= (s^2+1.84776S+1)(s^2+0.76536S+1)$$

$\therefore$ for 4th order Butterworth filter T.F is ($w_c=1$ rad/sec)

$$H(S) = \frac{1}{(s^2+0.76536S+1)(s^2+1.84776S+1)}$$

Similarly The Butterworth polynomials for various values of N for $w_c = 1$ rad/sec given by

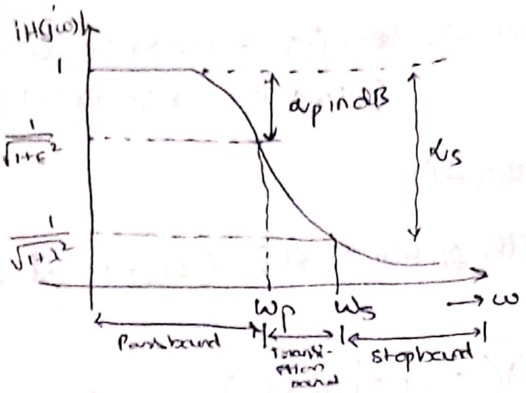
N	Denominator of H(S)
1	$S+1$
2	$S^2+\sqrt{2}S+1$
3	$(S+1)(S^2+S+1)$
4	$(s^2+0.7657S+1)(s^2+1.8477S+1)$
5	$(S+1)(S^2+0.61803S+1)(s^2+1.61803S+1)$
6	$(S^2+1.931855S+1)(s^2+\sqrt{2}S+1)(s^2+0.51764S+1)$
7.	$(S+1)(s^2+1.80194S+1)(s^2+1.247S+1)(s^2+0.445S+1)$

Eq 6 represents pole location of Butterworth filter for $w_c=1$ rad/sec. and are known as Normalised poles.

In General unNormalised poles are given by $S_K' = w_c S_K$

* The Transfer function of such type of Butterworth filter can be obtained by substituting $s \rightarrow S/w_c$ in T.F of Butterworth filter as shown in above table.

DERIVATION FOR ORDER OF FILTER :-



$\varepsilon \rightarrow$ parameter Specifying allowable Pass band

$\lambda \rightarrow$ Parameter Specifying allowal stopband

$\omega_p \rightarrow$ Passband frequency.

$\alpha_p \rightarrow$ Passband Attenuation.

$\omega_s \rightarrow$ Stopband frequency.

$\alpha_s \rightarrow$ Stopband Attenuation.

Let maximum passband attenuation α_p (< 3dB) at passband frequency ω_p and α_s is minimum Stopband Attenuation at stopband frequency ω_s.

Now magnitude of function can be written as

$$|H(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2(\omega/\omega_p)^{2N}}} \quad \text{———} \quad (1)$$

$$\Rightarrow |H(j\omega)|^2 = \frac{1}{1+\varepsilon^2(\omega/\omega_p)^{2N}}$$

Applying 10log on both sides.

$$20\log|H(j\omega)| = 10\log 1 - 10\log\left(1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}\right) \quad \text{———} \quad (2)$$

from figure we have $\omega = \omega_p$, Attenuation is α_p

$\therefore$ from Eq(2) $20\log|H(j\omega)| = -10\log(1+\varepsilon^2) = -\alpha_p$

$$\Rightarrow \alpha_p = 10\log(1+\varepsilon^2).$$

$$\Rightarrow \varepsilon = \sqrt{(10^{0.1\alpha_p}-1)} \quad \text{———} \quad (3)$$

at $\omega = \omega_s$, minimum stopband attenuation is α_s

from (2) $20\log|H(j\omega)| = 10\log 1 - 10\log\left(1+\varepsilon^2\left(\frac{\omega_s}{\omega_p}\right)^{2N}\right)$

$$\Rightarrow -\alpha_s = -10\log\left(1+\varepsilon^2\left(\frac{\omega_s}{\omega_p}\right)^{2N}\right)$$

$$\Rightarrow \left(\frac{\omega_s}{\omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{\varepsilon^2}$$

from (3) $\Rightarrow$

$$\left(\frac{\omega_s}{\omega_p}\right)^{N} = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} \quad \text{———} \quad (4)$$

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Applying log value on bothsides, we get (5)

$$\log\left[\left(\frac{\omega s}{\omega p}\right)^N\right] = \log\left(\sqrt{\frac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}}\right)$$

$$\Rightarrow \quad N = \frac{\log\left(\sqrt{\frac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}}\right)}{\log\left(\frac{\omega s}{\omega p}\right)} \quad \text{———} \; (5)$$

If above Expression doesnot gives any integer value, N will be roundoff to Next higher integer.

i.e. $$N \geq \frac{\log\left(\sqrt{\frac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}}\right)}{\log\left(\frac{\omega s}{\omega p}\right)}$$

$\frac{\omega p}{\omega s} = K$

$$\Rightarrow \boxed{N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(1/K\right)}} \quad \text{———} \; (6)$$

where $\varepsilon = \sqrt{10^{0.1\alpha p}-1}$, $\lambda = \sqrt{10^{0.1\alpha s}-1}$ and $K = \frac{\omega p}{\omega s}$.

(6) $\Rightarrow \boxed{N \geq \frac{\log A}{\log(1/K)}}$ where $A = \frac{\lambda}{\varepsilon}$.

Steps to design Analog Butterworth LPF

1. from The given Specifications, Find Order of filter 'N'.
2. Round off it to The next higher integer
3. Find Transfer function H(s) for $\omega c = 1$ rad/sec for the value of 'N'.
4. calculate ωc value, $\omega c = \frac{\omega p}{(10^{0.1\alpha p}-1)^{1/2N}}$
5. Find Transfer function $H_a(s)$ for above value of ωc by Substituting $s \to \frac{s}{\omega c}$ in H(s).

Ex: Design an analog Butterworth filter That has a $\alpha p = 2$dB at $\Omega p = 20$ rad/sec, $\alpha s = 10$ dB at $\Omega s = 30$ rad/sec

Problems:

Ex: Let $\alpha_p = 1\,dB$, $\alpha_s = 30\,dB$, $\omega_p = 200$ rad/sec, $\omega_s = 60$

Then find order of filter.

Sol:

We know

$$N \geq \frac{\log A}{\log(1/k)}$$

where $A = \frac{\lambda}{\varepsilon} = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} = \sqrt{\frac{10^3-1}{10^{0.1}-1}} = 62.115$

& $k = \frac{\omega_p}{\omega_s} = \frac{200}{600} = 1/3$

$$\therefore \quad N \geq \frac{\log(62.115)}{\log(3)} = 3.758$$

Roundoff N to Next higher integer we get $N = 4$.

Ex: Design an analog Butterworth filter that has a $-2\,dB$ passband attenuation at 20r frequency and $-10\,dB$ stopband attenuation at 30 rad/sec.

Sol: Let $\alpha_p = 2\,dB$; $\omega_p = 20$ rad/sec

$\alpha_s = 10\,dB$; $\omega_s = 30$ rad/sec

$$N \geq \frac{\log A}{\log(1/k)} = \frac{\log\left(\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}\right)}{\log\left(\frac{\omega_s}{\omega_p}\right)} \geq 3.37$$

$\therefore N = 4$

We have the Normalise butterworth filter N=4 has

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

But we know $\omega_c = \frac{\omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{20}{(10^{0.2}-1)^{1/8}} = 21.3688$

The Transfer function for $\omega_c = 21.3688$ can be obtained by letting $s \to \frac{s}{\omega_c}$ in H(s)

$$\Rightarrow H(s) = \frac{1}{\left[\left(\frac{s}{21.388}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right)+1\right]} \times \frac{1}{\left[\left(\frac{s}{21.388}\right)^2 + \frac{1.8477s}{21.3861} \cdots\right.}$$

$$= \frac{0.20921 \times 10^6}{(s^2+16.3686s+457.394)(s^2+39.5176s+457\cdots}$$

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Ex: Prove That $W_c = \frac{\omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$

Sol: The magnitude square function of Butterworth analog LPF is given by $|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2N}}$ —— (1)

$$\Rightarrow |H(j\omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}} \quad \text{—— (2)}$$

Equating (1) & (2)

$$\Rightarrow 1+\left(\frac{\omega}{\omega_c}\right)^{2N} = 1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N} \Rightarrow \left(\frac{\omega}{\omega_c}\right)^{2N} = \varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}$$

$$\Rightarrow \left(\frac{\omega_p}{W_c}\right)^{2N} = 10^{0.1\alpha_p}-1$$

$$\rightarrow \boxed{W_c = \frac{\omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}}} = \frac{\omega_p}{\varepsilon^{1/N}} \quad \text{—— (3)}$$

But we know $\left(\frac{W_s}{\omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$

$$\Rightarrow \omega_s = \omega_p\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2N} \Rightarrow \omega_s = W_c\cdot\cancel{(10^{0.1\alpha_p}-1)^{1/2N}}\left[\frac{10^{0.1\alpha_s}-1}{\cancel{10^{0.1\alpha_p}-1}}\right]^{1/2N}$$

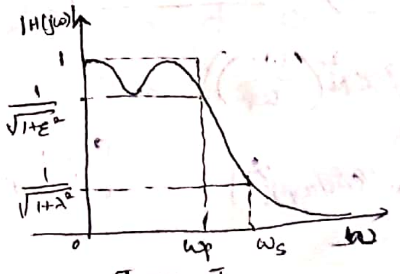
$$\Rightarrow \omega_s = W_c\,(10^{0.1\alpha_s}-1)^{1/2N}$$

$$\therefore \boxed{\omega_c = \frac{\omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}} = \frac{W_s}{\lambda^{1/N}} \quad \text{—— (4)}$$

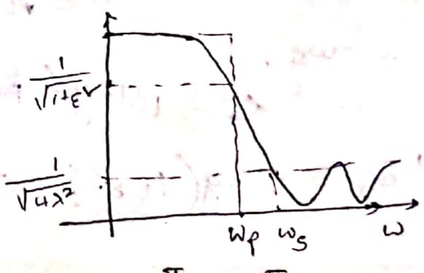
ANALOG LOWPASS CHEBYSHEV FILTER:

There are Two types of chebyshev filters.

* Type-I chebyshev filters are all-pole filters. That exhibits equi~~[illegible]~~ ripple (small wave) behaviour at pass-band and a Monotonic characteristics in stopband
* Type-II chebyshev filters are having both poles & zeros and exhibits a Monotonic (without any change) behaviour in Passband and an Equiripple behaviour in stopband



Type - I



Type - II

Type-I chebyshev filter :

The Magnitude square response of N^{th} order type-I chebyshev filter can be expressed as

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\omega}{\omega_p}\right)} \quad N=1,2 \ldots \quad \text{—— (1)}$$

$\varepsilon \rightarrow$ Parameter of filter related to ripple in Passband and $C_N(x)$ is N^{th} order chebyshev polynomial defined as

$$\left.\begin{aligned} C_N(x) &= \cos(N\cos^{-1}(x)),\ |x|\le 1 \ (\text{Passband}) \\ C_N(x) &= \cosh(N\cosh^{-1}x),\ |x|>1 \ (\text{stopband}) \end{aligned}\right\} \text{and} \quad \text{—— (2)}$$

These chebyshev polynomials is defined by Recursive formula. $C_N(x) = 2x C_{N-1}(x) - C_{N-2}(x),\ N>1$ —— (3)

The polynomials can follow the following Properties

1. $C_N(x) = -C_N(-x)$ for N-odd
   $C_N(x) = C_N(-x)$ for N-Even
   $C_N(0) = (-1)^{N/2}$ for N - Even
   $C_N(0) = 0$ for N-odd
   $C_N(1) = 1$ for all N
   $C_N(-1) = 1$ for N-Even
   $C_N(-1) = -1$ for N-odd.
2. $C_N(x)$ is monotonically increasing for $|x|>1$ for all N.
3. $C_N(x)$ Oscillates with Equal ripple b/w ± 1 for $|x|\le 1$
4) for all N $0 \le |C_N(x)| \le 1$ for $0 \le |x| \le 1$
   $|C_N(x)| > 1$ for $|x|>1$

Apply dB value to eq (1)

$$20\log|H(j\omega)| = 10\log 1 - 10\log\left(1+\varepsilon^2 C_N^2\left(\frac{\omega}{\omega_p}\right)\right)$$

let α_p — passband Attenuation
α_s — Stopband Attenuation

$\therefore$ at $\omega=\omega_p$, $\alpha_p = 10\log(1+\varepsilon^2)$ $[\because C_N(1)=1]$

$$\rightarrow \varepsilon = \sqrt{10^{0.1\alpha_p}-1} \quad \text{—— (4)}$$

at $\omega=\omega_s$,

$$\alpha_s = 10\log\left(1+\varepsilon^2 \cdot C_N^2\left(\frac{\omega_s}{\omega_p}\right)\right)$$

$$\Rightarrow \alpha_s = 10\log\left(1+\varepsilon^2\{\cosh(N\cosh^{-1}(\omega_s/\omega_p))\}^2\right] \quad \left[\because \frac{\omega_s}{\omega_p}>1\right]$$

$$\Rightarrow (10^{0.1\alpha_s}-1) = \varepsilon^2 \{\cosh[N \cdot \cosh^{-1}(\omega_s/\omega_p)]\}^2$$

$$\Rightarrow \cosh(N \cdot \cosh^{-1}(\omega_s/\omega_p)) = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}$$

$$\Rightarrow N \geq \frac{\cosh^{-1}\left(\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}\right)}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)}$$

[$\because$ N round off next hightest integer value]

$$\Rightarrow N \geq \frac{\cosh^{-1}A}{\cosh^{-1}(1/k)} \longrightarrow (5)$$

here $\cosh^{-1}(x)$ can be Evaluated using Identity

$$\cosh^{-1}x = \ln[x+\sqrt{x^2-1}] \longrightarrow (6)$$

POLE LOCATION FOR CHEBYSHEV FILTER :

The poles for type-I filter are obtained by Equating The denominator value to zero and let $w = -js$.

$$\Rightarrow 1+\varepsilon^2 C_N^2\left(\frac{\omega}{\omega_p}\right) = 0$$

$$\Rightarrow 1+\varepsilon^2 C_N^2\left(\frac{-js}{\omega p}\right) = 0 \Rightarrow \varepsilon^2 C_N^2\left(\frac{-js}{\omega p}\right) = -1$$

$$\Rightarrow C_N\left(\frac{-js}{\omega p}\right) = \pm\frac{j}{\varepsilon}$$

for Pass band The polynomial $C_N(x) = \cos(N\cos^{-1}(x))$

$$\text{so } C_N\left(\frac{-js}{\omega p}\right) = \cos\left(N\cos^{-1}\left(\frac{-js}{\omega p}\right)\right) = \pm\frac{j}{\varepsilon}$$

$$\Rightarrow \text{let } \cos^{-1}\left(\frac{-js}{\omega p}\right) = \phi - j\theta \longrightarrow (1)$$

$$\Rightarrow \pm\frac{j}{\varepsilon} = \cos(N\phi - jN\theta)$$

$$\Rightarrow \pm\frac{j}{\varepsilon} = \cos N\phi \cos(jN\theta) + \sin(N\phi)\cdot\sin(jN\theta)$$

$$= \cos(N\phi)\cosh(N\theta) + j\sin(N\phi)\sinh N\theta \quad (2)$$

We know

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\cos j\theta = \frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2} = \frac{e^{-\theta}+e^{\theta}}{2} = \cosh\theta$$

Equating Real & imaginary terms.

$$\left.\begin{array}{ll} \cos(N\phi)\cdot\cosh(N\theta) = 0 & \longrightarrow (3a) \\ \sin N\phi \cdot \sinh N\theta = \pm\frac{1}{\varepsilon} & \longrightarrow (3b) \end{array}\right\} (3)$$

from Eq (3a) let $\cosh(N\theta) \neq 0$ for a real value

$$\text{so } \cos N\phi = 0 \Rightarrow \phi = \frac{(2k-1)\pi}{2N}, \; k=1,2\cdots N \longrightarrow (4a)$$

$$\text{from Eq (3b) } \sinh(N\theta) = \pm 1/\varepsilon \Rightarrow \theta = \pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \longrightarrow (4b)$$

(4)

∴ by Combining Eq ① & ②, 4a, 4b. we get

$$\cos(\phi - j\theta) = \frac{-js}{\omega p}$$

$$\Rightarrow s = \frac{j\omega p}{-j} \cos(\phi - j\theta) \Rightarrow S_K = j\omega p \cos(\phi - j\theta)$$

$$\Rightarrow S_K = j\omega p[\cos\phi \cosh\theta + j\sin\phi \cdot \sinh\theta] \quad \text{—— ⑤}$$

$$= \omega p[-\sin\phi \cdot \sinh\theta + j\cos\phi \cdot \cosh\theta]$$

Eq ⑤ can be simplified by having identity

$$\sinh^{-1}(\varepsilon^{-1}) = \ln(\varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}) \quad [\because \sinh^{-1}x = \ln(x + \sqrt{1+x^2})]$$

$$\Rightarrow \text{let } \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = e^{\sinh^{-1}(\varepsilon^{-1})} = \mu. \quad \text{—— ⑥}$$

We know $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\Rightarrow \sinh\theta = \sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{e^{\{\frac{1}{N}\sinh^{-1}(\frac{1}{\varepsilon})\}} - e^{-\{\frac{1}{N}\sinh^{-1}(\frac{1}{\varepsilon})\}}}{2}$$

$$= \frac{\left[e^{\{\sinh^{-1}(\frac{1}{\varepsilon})\}}\right]^{1/N} - \left[e^{-\{\sinh^{-1}(\frac{1}{\varepsilon})\}}\right]^{1/N}}{2}$$

$$= \frac{\mu^{1/N} - \mu^{-1/N}}{2} \quad \text{—— ⑦}$$

Similarly $\cosh\theta = \frac{\mu^{1/N} + \mu^{-1/N}}{2}$ —— ⑧

∴ from Eq ⑤

$$S_K = \omega p\left\{-\sin\phi\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] + j\cos\phi\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]\right\}$$

$$= -a\sin\phi + j\cos\phi$$

$$= -a\sin\left(\frac{(2K-1)\pi}{2N}\right) + jb\cos\left(\frac{(2K-1)\pi}{2N}\right)$$

$$= a\cos\left(\frac{\pi}{2} + \frac{(2K-1)\pi}{2N}\right) + jb\sin\left[\frac{\pi}{2} + \frac{(2K-1)\pi}{2N}\right]$$

$$= a\cos\phi_K + jb\sin\phi_K \quad \text{—— ⑨}$$

$$= \sigma_K + j\omega_K, \quad K = 1, 2, \ldots N.$$

∴ The poles of chebyshev filter can be calculated from Eq ⑨

8

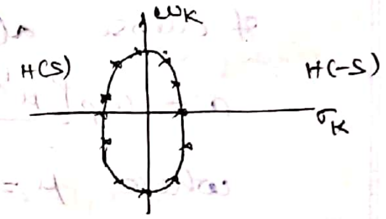
where $a = \omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$ where $\mu = e^{\sinh^{-1}(\varepsilon^{-1})}$

$b = \omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$ $\quad = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$.

$$\phi_K = \frac{\pi}{2} + \frac{(2K-1)\pi}{2N}, \; K = 1, 2 \cdots N.$$ ~~and~~

The poles of chebyshev Transfer function are located on an ellipse in the s-plane where Eq of Ellipse is $\frac{\sigma_K^2}{a^2} + \frac{\omega_K^2}{b^2} = 1$

where a, b are ~~major~~ minor and major axis of Ellipse respectively.



CHEBYSHEB-TYPE-II FILTER:

chebyshev type-II filter has both poles and zeros. The magnitude Response is given by

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2\left[\frac{C_N^2\left(\frac{\omega_s}{\omega_p}\right)}{C_N^2\left(\frac{\omega_s}{\omega}\right)}\right]}$$

let $C_N(x)$ is N^{th} order chebyshev polynomial, ω_s, ω_p -

* The zeros located at imaginary axis at points $S_K = j\frac{\omega_s}{\sin\phi_K}, K = 1, 2 \cdots N$
* The poles located at points (x_K, y_K) where

$$x_K = \frac{\omega_s \sigma_K}{\sigma_K^2 + \omega_K^2}, \; K = 1, 2 \cdots N$$

$$y_K = \frac{\omega_s \cdot \omega_K}{\sigma_K^2 + \omega_K^2}, \; K = 1, 2 \cdots N$$

where $\sigma_K = a\cos\phi_K$ and $\mu = \lambda + \sqrt{1+\lambda^2}$

$\omega_K = b\sin\phi_K$

for given specifications order of the filter is given by $N = \frac{\cosh^{-1}\left(\frac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)} = \frac{\cosh^{-1}(A)}{\cosh^{-1}(1/k)}$

where $A = \lambda/\varepsilon$, $k = \frac{\omega_p}{\omega_s}$.

$\varepsilon = \sqrt{(10^{0.1\alpha_p} - 1)}^{0.5}$ & $\lambda = \sqrt{10^{0.1\alpha_s} - 1}$

Steps to design an Analog chebyshev LPF

1. From given specifications find order of filter
2. Round off it to its next heighest Integer
3. using following Relations find major and minor a of Ellipse a(minor)

$$a = \omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right], \quad b = \omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where $\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$, $\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$

ω_p — Pass band frequency,

α_p — maximum allowable attenuation in Passban

(For Normalised chebyshev filter let $\omega_p = 1$)

4. find poles which are lies on Ellipse by the formu

$$S_k = a\cos\phi_k + jb\sin\phi_k, \quad k = 1, 2 \ldots N.$$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2 \ldots N$.

5. find Denominator Polynomial using above pol
6. Numerator of Transfer function depends on N va
   a) for N-odd, take $s=0$ in denominator Polynom and find value. This is Equal to Numerator Top (for N-od $|H(j\omega)|$ starts at 1)
   b) for N-Even let $s=0$ in denominator polyno and divide result by $\sqrt{1+\varepsilon^2}$. This value to Numerator.

Ex: Design a chebyshev filter with $\alpha_p = 2.5dB$ at $\omega_p = 20$ rad/sec and $\alpha_s = 30dB$ at $\omega_s = 50$ rad/sec.

Sol:

step1:
$$N = \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1}(1/k)}, \quad \lambda = \sqrt{10^{0.1\alpha_s} - 1} = 31.607$$
$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.882$$
$$k = 0.4 = \frac{\omega_p}{\omega_s}$$
$$\Rightarrow N \geq \frac{\cosh^{-1}\left(\frac{31.607}{0.882}\right)}{\cosh^{-1}\left(\frac{1}{0.4}\right)} = 2.726 \approx 3$$
$$\Rightarrow \boxed{N = 3}$$

step 2:

step 3:
$$\mu = \varepsilon^{-1} + \sqrt{1 - \varepsilon^{-2}} = 2.65$$
$$a = \omega_p\left(\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right) = 6.6, \quad b = \omega_p\left(\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right) = 21.06$$

step4:
$$s_k = a\cos\phi_k + jb\sin\phi_k, \quad k = 1, 2, 3$$
$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}, \quad k = 1, 2, 3 \dots$$
$$\phi_1 = 120°, \ \phi_2 = 180°, \ \phi_3 = 240°$$
$$s_1 = -3.3 + j18.23$$
$$s_2 = -6.6$$
$$s_3 = -3.3 - j18.23$$

$(s-s_1)(s-s_2)$ $s_1 s_2$

step5:
$$Den(H(s)) = (s + 6.6)(s^2 + 6.6s + 343.2)$$

step6: Numerator of $H(s) = (6.6)(343.2) = 2265.27$

Transfer function is
$$H(s) = \frac{2265.27}{(s+6.6)(s^2+6.6s+343.2)}$$

Ex: determine order and poles of chebyshev type-I LPF that has a 1dB ripple in passband at $\omega_p = 1000\pi$ and stopband freq of 2000π and an attenuation of 40dB or more.

Ans: order = 5

Poles $s_1 = -89.5\pi + j989\pi$; $s_4 = -234.2\pi - j612\pi$

$s_2 = -234.2\pi + j612\pi$; $s_5 = -89.5\pi - j989\pi$.

$s_3 = -289.5\pi$

Comparision between Butterworth and chebyshev filt

Butterworth	Chebyshev
1. The magnitude of this filter monotonically decreases as ω increases. from $0 \rightarrow \infty$.	1. The Magnitude respon of this filter exhibit ripples in Pass band. Stopband according Their types.
2. Transmission band is more	2. Transmission band less Compared to butterworth filter.
3. Poles are lies on Circle	3. Poles lies on Ellips
4. for Same Specifications In This number of Poles are more i.e Order of This filter is higher than chebyshev	4. for Same Specifica In This number of are less. i.e Order of This fil is Less. Than That of B-worth. This is great advantage becaus Less number of discret Components necessary Construct filter.

Frequency Transformation in Analog Domain.

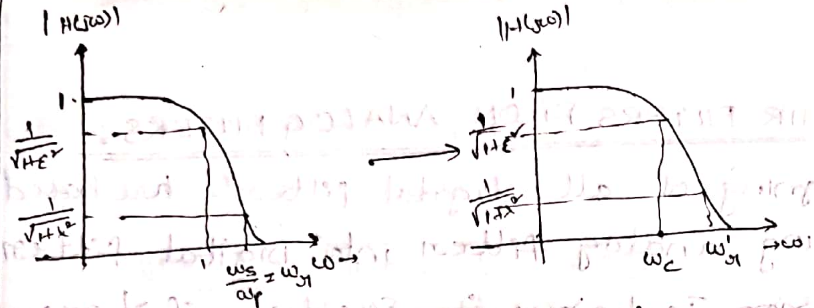
In This Session we will Discuss about designi of Lowpass, highpass, bandpass and Band reje filters from a Normalized LPF.

a) Low Pass to Low Pass:

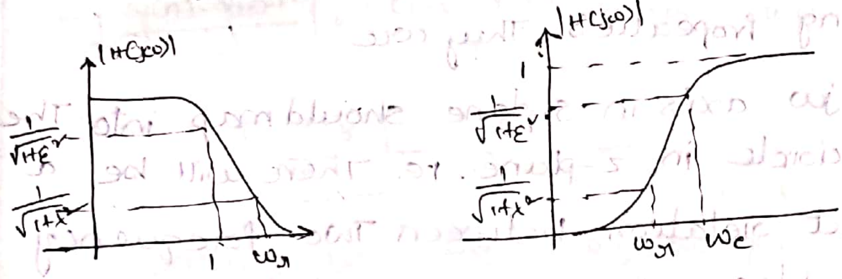
An unnormalized Lowpass filter must have a different cutoff frequency ω_c for This. we have a Transformation $S \longrightarrow \frac{S}{\omega_c}$

b) Lowpass to Highpass:

we have Transformation as $S \longrightarrow \frac{\omega_c}{S}$



Low Pass $\longrightarrow$ Low pass.



Low Pass $\longrightarrow$ High pass

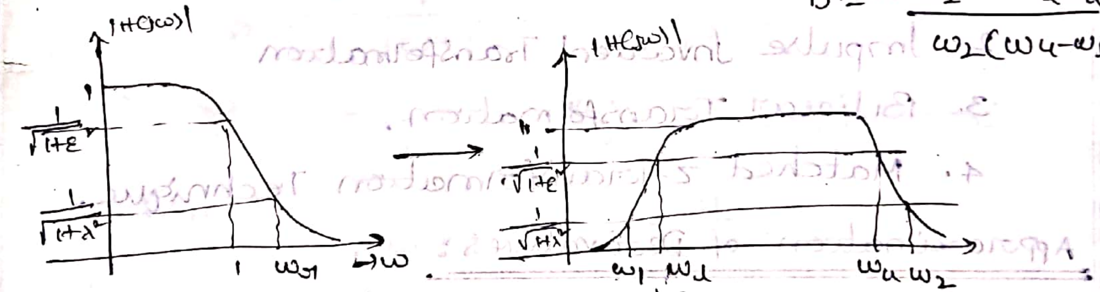
Low Pass to Band Pass filter

We have in band pass filters the cutoff frequences. ω_l, ω_u. Can be accomplished by

$$s \longrightarrow \frac{s^2+\omega_l\omega_u}{s(\omega_u-\omega_l)}$$

$$\omega_r = \min\{|A|,|B|\}$$ where $A = \dfrac{-\omega_1^2+\omega_l\omega_u}{\omega_1(\omega_u-\omega_l)}$

$$B = \frac{\omega_2^2-\omega_l\omega_u}{\omega_2(\omega_u-\omega_l)}$$



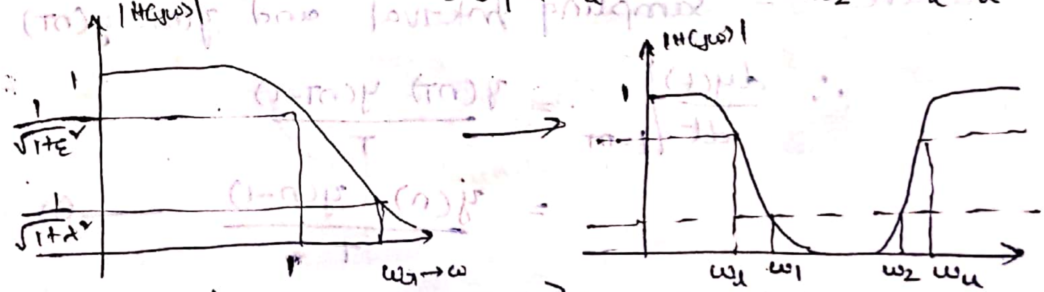
Low Pass $\longrightarrow$ Band Pass

Low Pass to Band Stop filter:

We have Transformation as $s \longrightarrow \dfrac{s(\omega_u-\omega_l)}{s^2+\omega_u\omega_l}$

$$\omega_r = \min\{|A|,|B|\}$$

where $A = \dfrac{\omega_1(\omega_u-\omega_l)}{-\omega_1^2+\omega_l\omega_u}$, $B = \dfrac{\omega_2(\omega_u-\omega_l)}{-\omega_2^2+\omega_l\omega_u}$



lowpass $\longrightarrow$ Bandstop

DESIGN OF IIR FILTERS FROM ANALOG FILTERS:

The designing of all digital filters are based on converting analog filter into Digital filter. The conversion Technique is Effective if it posses the following properties. They are

(i) The $j\omega$ axis in s-plane should map into the unit circle in z-plane. i.e. There will be a direct relation between two frequency variables in the two domains.

(ii) The left half of the s-plane should map into the inside of unit circle in z-plane i.e a stable Analog filter is converted into stable digital filter

So to digitize Analog filter into a digital filter we have 4 Methods. They are.

1. Approximation of Derivatives.
2. Impulse Invarient Transformation
3. Bilinear Transformation.
4. Matched z-Transformation Technique.

Approximation of Derivatives:

This is simplest Method of digitizing an analog filter into a digital filter is to approximate the differential Equation by an Equivalent difference Eq.

for derivative $\frac{dy(t)}{dt}$ at time $t = nT$ we have a difference Equation $\frac{y(nT) - y(nT-1)}{T}$

where T - sampling Interval and $y(n) = y(nT)$

$$\therefore \left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-1)}{T}$$

$$= \frac{y(n) - y(n-1)}{T} \quad \text{——(1)}$$

But we know $L\left[\frac{dy(t)}{dt}\right] = S \cdot Y(S)$ —— (2)

Eq ①, ② are represented as

$$y(t) \longrightarrow \boxed{H(S) = S} \longrightarrow \frac{dy(t)}{dt}$$

$$Z\left[\frac{y(n)-y(n-1)}{T}\right] = \frac{(1-z^{-1})Y(z)}{T}$$ can be represented as

$$y(t) \longrightarrow \boxed{\frac{(1-z^{-1})}{T}} \longrightarrow \frac{y(n)-y(n-1)}{T}$$

Comparing both figures. we get

$$S = \frac{1-z^{-1}}{T} \quad \text{——} ③$$

∴ The system function for IIR digital filter obtained as a result of approximations of derivatives by finite difference is $H(z) = H(S)\Big|_{S = \frac{1-z^{-1}}{T}}$

from Eq ③ $\quad S = \frac{1-z^{-1}}{T} \Rightarrow z = \frac{1}{1-ST}$ but we know

$S = j\omega$

$$\Rightarrow z = \frac{1}{1-j\omega T} = \frac{1+j\omega T}{1+\omega^2 T^2}$$

$$\Rightarrow z = \frac{1}{1+\omega^2 T^2} + \frac{j\omega T}{1+\omega^2 T^2} = x + jy \quad \text{——} ④$$

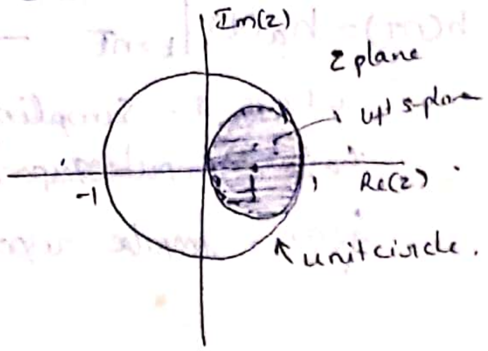
where $x = \frac{1}{1+\omega^2 T^2}$, $y = \frac{\omega T}{1+\omega^2 T^2}$

x, y are represented by a relation of $x^2+y^2 = x$ —— ⑤

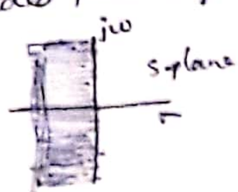
which can be represented as.

$$\left(x-\tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4} \quad \text{——} ⑥$$

∴ Image in z-plane of $j\omega$ in s-plane is of $\frac{1}{2}$ radius as shown below.



shaded parts represents s-plane



from figure we observed That

1. The left half S-plane maps inside a circle of radius $\frac{1}{2}$ centered at $z = \frac{1}{2}$ in z-plane
2. The Right half s-plane maps into outside of circle of Radius $1/2$ in z-plane
3. The $j\omega$-maps on perimeter of the circle of radius $\frac{1}{2}$ in z-plane.

There is some restrictions to convert analog LPF → Digital LPF, lly HPF → HPF, BPF − BPF where (have small resona

Matched Z-Transform Method:

Another method for converting an analog filter into an Equivalent digital filter is to map The Poles and zeros of H(s) directly into Poles and zeros in z-plane.

If

$$H(s) = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - P_k)}$$

where $\{z_k\}$ → zeros, $\{P_k\}$ → Poles. of filter

Then System function of digital filter is

$$H(z) = \frac{\prod_{k=1}^{M}(1 - e^{z_k T} z^{-1})}{\prod_{k=1}^{N}(1 - e^{P_k T} \cdot z^{-1})}$$

where T - Sampling interval

i.e Each factor of the form $(s-a)$ in $H(s)$ is mapped into the factor $1 - e^{aT} z^{-1}$. This mapping is called Matched Z-Transform.

Impulse Invarient Transformations:

The Impulse response of the digital filter is obtained by uniformly sampling the impulse response of the Analog filter.

$$\text{i.e} \quad h(n) = h(nT) = h_a(t)\Big|_{t=nT} \qquad \text{--- (1)}$$

where T - Sampling Interval

$h_a(t)$ - Impulse response of Analog Filter

$h(n)$ - Impulse response of Digital filter let us consider the Transfer function of Analog filter which contains distinct poles.

$$H_a(s) = \sum_{K=1}^{N} \frac{A_K}{s-P_K} \quad \text{——— (2)}$$

The impulse response of the above Equation is given as

$$h_a(t) = \sum_{K=1}^{N} A_K \cdot e^{P_K t} \cdot u_a(t) \quad \text{——— (3)}$$

where $u_a(t)$ — continuous time step function.

Then $h(n)$ can be obtained by Eq (1)

$$h(n) = h_a(nT) = \sum_{K=1}^{N} A_K e^{nP_KT} \cdot u_a(nT)$$

The system function $H(z)$ of digital filter is ~~filter~~ given by

$$H(z) = Z\{h(n)\} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$\Rightarrow H(z) = Z\left\{\sum_{K=1}^{N} A_K \cdot e^{nP_KT}\right\}$$

$$= \sum_{K=1}^{N} A_K \cdot Z\{e^{nP_KT}\}$$

$$= \sum_{K=1}^{N} \frac{A_K}{1-e^{P_KT} \cdot z^{-1}} \quad \text{——— (4)}$$

Comparing Eqs (2) & (4) we get $\frac{1}{s-P_K} = \frac{1}{1-e^{P_KT} \cdot z^{-1}}$

$$\Rightarrow \frac{1}{s-P_K} = \frac{z}{z-e^{P_KT}} \quad \text{——— (5)}$$

from Eq (5) we get analog pole at $s=P_K$ is mapped onto a digital pole at $z=e^{P_KT} = e^{ST} \Rightarrow z = e^{ST}$.

we know $s = \sigma + j\omega$ and $z = re^{jW}$

$$\Rightarrow re^{jW} = e^{(\sigma+j\omega)T}$$

from above $r = e^{\sigma T}$ ——— (6a) and $W = \omega T$ ——— (6b).

Eq (6b) gives the Relation between analog frequency ω and digital frequency (W) in Impulse Invariant method.

from Equation (6a) if $\sigma < 0$, r lies b/w 0 and 1.

if $\sigma = 0$, $r = 1$

if $\sigma > 0$, $r > 1$

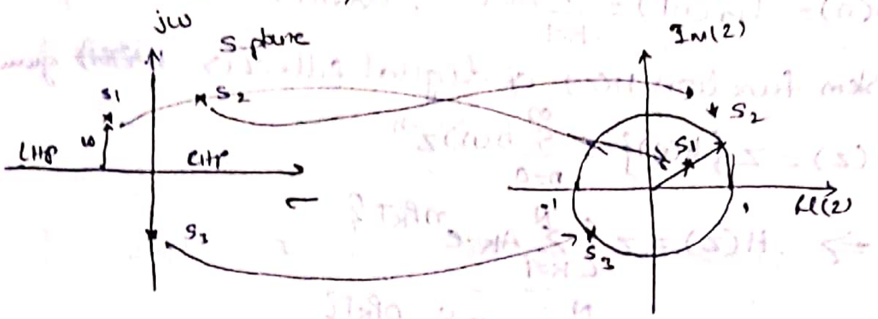
from above results we can state that :

(i) The Poles on the left half of the s-plane will be mapped into the inside of the unit circle in z-plane. Thus stable digital filter is obtained.

(ii) Poles lies on the jω-axis in s-plane will be mapped on to the unit circle in z-plane

(iii) The Poles lies on Right half of s-plane will be mapped into the outside of unit circle in z-plane.

Below figure represents the mapping s-plane into z-plane.



This mapping is not one-to-one rather it is many to one, i.e. many points in the splane are mapped to a single point in the z-plane. To show this let us consider Two poles in 's'plane with Identical real parts, But Imaginary parts differ by $\frac{2\pi}{T}$

Let Two poles $s_1 = \sigma + j\omega$

$s_2 = \sigma + j\left(\omega + \frac{2\pi}{T}\right)$

These are mapped to z-plane poles of z_1, z_2

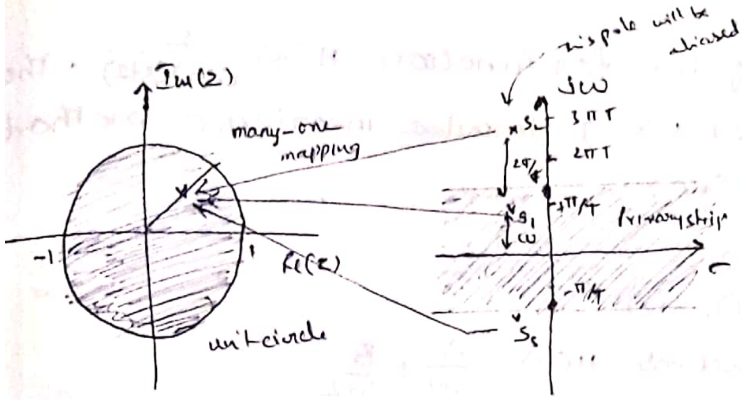
$$z_1 = e^{(\sigma + j\omega)T} = e^{\sigma T} \cdot e^{j\omega T}$$

$$z_2 = e^{\left\{\sigma + j\left(\omega + \frac{2\pi}{T}\right)\right\}T} = e^{\sigma T} \cdot e^{j\omega T} \cdot e^{j2\pi} \quad \left\{\because e^{j2\pi} = 1\right\}$$

$$\Rightarrow z_2 = e^{\sigma T} \cdot e^{j\omega T}$$

i.e $z_1 = z_2$

it is clear That These poles map to the same location in z-plane. There are infinite number of s-plane poles (which have same real parts and imaginary parts differ by an integer multiple of $\frac{2\pi}{T}$) that map to same locations in z-plane



This is The biggest disadvantage of impulse-invarient method. The s-plane poles having imaginary Parts greater Than $\frac{\pi}{T}$ or less Than $-\frac{\pi}{T}$ causes aliasing

The analog poles will not be aliased by The Impulse invarient mapping if They are confined to s-plane's "Primary strip" (within π/T of The real axis)

Due to The Presence of aliasing, This Method is appropriate for The design of Lowpass and band pass filters only. This Method is unsuccessful for designing high pass filter.

Ex: for analog Transfer function $H(s) = \frac{2}{(s+1)(s+2)}$, determine $H(z)$ using impulse invariance Method. Assume $T = 1$ sec.

Sol: given $H(s) = \frac{2}{(s+1)(s+2)}$

steps to design a digital filter using IIM.

1. For The given specifications find $H_a(s)$. i.e. Transfer function of Analog filter
2. select Sampling Rate of digital filter, T seconds/sample
3. express Analog filter T.F as sum of single pole filters

$$H_a(s) = \sum_{K=1}^{N} \frac{c_K}{s - P_K}$$

4. Compute z-Transform of digital filter by using formula

$$H(z) = \sum_{K=1}^{N} \frac{c_K}{1 - e^{P_K T} z^{-1}}$$

for High sampling Rates

$$H(z) = \sum_{K=1}^{N} \frac{T c_K}{1 - e^{P_K T} \cdot z^{-1}}$$

Ex: For an Analog Transfer function $H(s)=\frac{2}{(s+1)(s+2)}$. Then determine $H(z)$ using impulse invarience method, assume $T=1$ sec.

Sol: $H(s)=\frac{2}{(s+1)(s+2)}$

using partial fractions. $H(s)=\frac{A}{s+1}+\frac{B}{s+2}$

By inspection $A=2$, $B=-2$

$\therefore H(s)=\frac{2}{s+1}-\frac{2}{s+2} \Rightarrow H(s)=\frac{2}{s-(-1)}-\frac{2}{s-(-2)}$

using impulse invarient Technique we have, if

$H(s)=\sum_{K=1}^{N}\frac{C_K}{s-P_K}$ Then $H(z)=\sum_{K=1}^{N}\frac{C_K}{1-e^{P_K T}.z^{-1}}$

i.e $(s=P_K)$ is Transformed To $1-e^{P_K T}.z^{-1}$

we have, Two poles $P_1=-1$, $P_2=-2$.

$\therefore H(z)=\frac{2}{1-e^{-T}z^{-1}}-\frac{2}{1-e^{-2T}.z^{-1}}$

given T=1 sec

$$\therefore H(z)=\frac{2}{1-e^{-1}z^{-1}}-\frac{2}{1-e^{-2}z^{-1}}$$

$$=\frac{2}{1-0.3678z^{-1}}-\frac{2}{1-0.1353z^{-1}}$$

$$\therefore H(z)=\frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}.$$

Ex: using impulse invariente with T=1 sec determine $H(z)$ if $H(s)=\frac{1}{s^2+\sqrt{2}s+1}$

steps:

$$H(s)=\frac{1}{(s+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}=\sqrt{2}.\frac{1/\sqrt{2}}{(s+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}$$

$$\rightarrow L^{-1}[H(s)]=\sqrt{2}L^{-1}\left[\frac{1/\sqrt{2}}{(s+\frac{1}{\sqrt{2}})^2+(\frac{1}{\sqrt{2}})^2}\right]=\sqrt{2}.e^{-t/\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right).$$

Let $t=nT$

$$\Rightarrow h(n)=\sqrt{2}.e^{-nT/\sqrt{2}}\sin\left(\frac{nT}{\sqrt{2}}\right).$$

$$z[h(n)]=H(z)=\sqrt{2}\left[\frac{e^{-1/\sqrt{2}}.z^{-1}.\sin 1/\sqrt{2}}{1-2e^{-1/\sqrt{2}}z^{-1}.\cos\frac{1}{\sqrt{2}}+e^{-\sqrt{2}}z^{-2}}\right].$$

$$H(z) = \frac{0.4532z^{-1}}{1-0.7497z^{-1}+0.2432z^{-2}}$$

This is the Required digital system function.

(iv) THE BILINEAR TRNSFORMATION METHOD:

The IIR digital filter design using IIT method is appropriate for designing LPF, BPF only and it is not suitable for Highpass or Bandstop filters. This limitation will be due to 'Aliasing Effect.' This limitation will be overcome by a Mapping Technique is called "Bilinear Transformation" Method. This is a one-one mapping from s-Domain to z-domain i.e. we Transforms jω-axis into unit circle in z-plane only once. Thus avoiding aliasing of frequency Components. All points in LHP of 's' are mapped inside the unit circle in z-plane and all Points in RHP of 's' are mapped into corresponding Points outside the unit circle in z-plane.

Consider the first order differential Equation which represents analog system is

$$\frac{d\,y(t)}{dt} = x(t) \quad \text{———(1)}$$

Integrating eq(1) on both sides, we get

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt}\,dt = \int_{(n-1)T}^{nT} x(t)dt$$

$$\Rightarrow \left[y(t)\right]_{(n-1)T}^{nT} = \left[\frac{x^2(t)}{2}\right]_{(n-1)T}^{nT}$$

$$\Rightarrow y(nT) - y[(n-1)T] = \frac{1}{2}\left[x^2(nT) - x^2((n-1)T)\right]$$

But we know Trapezoidal Rule

$$x(t) = \int_{t_0}^{t} x'(t)dt + x(t_0)$$

$$\text{so} \quad y(nT) - y((n-1)T) = \frac{T}{2}\left[x(nT) + x((n-1)T)\right]$$

for discrete time system, above Equation can be written as

$$y(n) - y(n-1) = \frac{T}{2}[x(n) + x(n-1)]$$

Convert above Equations into Z-Domain

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}[X(z) + z^{-1}X(z)]$$

$$\Rightarrow Y(z)\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] = X(z) \quad \text{--- (2)}$$

Take Laplace Transform of Eq(1) is $s \cdot Y(s) = X(s)$ --- (3)

Comparing (2), (3) we get.

The relation between s and z-domains as

$$s \longrightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \quad \text{--- (4)}$$

But we know $s = \sigma + j\omega$, $z = re^{jW}$

$$\therefore s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \Rightarrow s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$\Rightarrow \sigma + j\omega = \frac{2}{T}\left[\frac{re^{jW}-1}{re^{jW}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r[\cos W + j\sin W] - 1}{r[\cos W + j\sin W] + 1}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos W - 1) + jr\sin W}{(r\cos W + 1) + jr\sin W}\right]\left[\frac{(r\cos W + 1) - jr\sin W}{r(\cos W + 1) - jr\sin W}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2 W - 1 + r^2\sin^2 W + j2r\sin W}{[r\cos W + 1]^2 + r^2\sin^2 W}\right]$$

$$= \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos W} + j\frac{2r\sin W}{1+r^2+2r\cos W}\right]$$

By equating real and Imaginary parts.

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos W}\right] \quad \text{--- (5a)}, \qquad \omega = \frac{2}{T}\left[\frac{2r\sin W}{1+r^2+2r\cos W}\right] \quad \text{(5b)}$$

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(15)

if $r \le 1$ then $\sigma < 0$ LHP of s maps inside unit circle in z-plane.

$r > 1$ then $\sigma > 0$ RHP of s maps in outside of unit circle in z-plane

$r = 1$ then $\sigma = 0$ and $\omega = \frac{2}{T}\left[\frac{\sin W}{1+\cos W}\right]$

$$\Rightarrow \omega = \frac{2}{T} \cdot \frac{2\sin\frac{W}{2} \cdot \cos\frac{W}{2}}{2\cos^2\frac{W}{2}}$$

$$\omega = \frac{2}{T}\tan\frac{W}{2} \quad \Rightarrow W = 2\tan^{-1}\frac{\omega T}{2} \quad \text{——— (6)}$$

Eq (6) Represents the bilinear Transform relation between ω and W

$$\omega = \frac{2}{T}\tan\frac{W}{2}$$

for small value of W $\Rightarrow$ $\omega = \frac{2}{T} \cdot \frac{W}{2} = \frac{W}{T}$

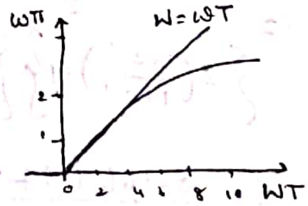
$$\Rightarrow \boxed{W = \omega \cdot T}$$

→ for low frequencies relation between ω, W is linear, hence digital filter has same magnitude response as that of analog filter.

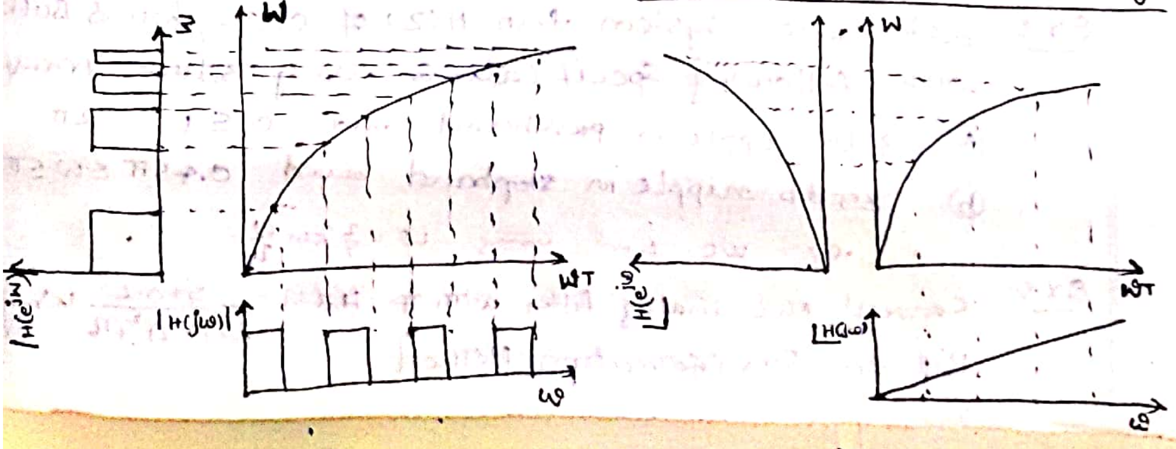
→ for high frequencies, the relation between W & ω becomes Non-linear and distortion is introduced in the frequency scale of the digital filter to that of analog filter. This is known as "Wrapping Effect".

The Influence of ~~wrapp~~ warping Effect on Magnitude and phase are represented as follows.

Relation b/w ω and W is



Effect on Magnitude due to warping   Effect on phase due to warping



Prewarping: The warping Effect can be Eliminated by Prewarping The Analog filter. This Can be done by finding Prewarping analog frequencies using The formula $\omega = \frac{2}{T} \tan \frac{W}{2}$.

i.e $\omega_p = \frac{2}{T} \tan \frac{W_p}{2}$.

$\omega_s = \frac{2}{T} \tan \frac{W_s}{2}$.

Steps to design digital filter using Bilinear Transform Tech

1) from Given Specifications, find Prewarping analog frequencies using $\omega = \frac{2}{T} \tan \frac{W}{2}$
2) Using Analog frequencies find H(s) of analog filter
3) select sampling rate of digital filter (T).
4) Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ into Transfer function in step 2.

Ex: apply Bilinear Transformation on $H(s) = \frac{2}{(s+1)(s+2)}$ with T=1 sec. and find H(z)

given, $H(s) = \frac{2}{(s+1)(s+2)}$

Put $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ in H(s) to get H(z)

$$\Rightarrow H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.332^{-1})}$$

Ex: determine system fun H(z) of Chebyshev & Butterworth with following Specifications, using Bilinear Transform

(a) 3 db ripple in passband and $0 \le W \le 0.2\pi$
(b) 25 db ripple in stopband and $0.45\pi \le W \le \pi$.

ω_p, W_s find using $\omega = \frac{2}{T} \tan \frac{W}{2}$

Ex: Convert the analog filter with TF $H_a(s) = \frac{s+0.2}{(s+0.2)^2+16}$ using Bilinear Transformation Method.

FREQUENCY TRANSFORMATION IN DIGITAL DOMAIN:

A digital LPF can be converted into a digital highpass, bandpass, bandstop or another digital filters. These Transformations given as follows.

(a) LOWPASS TO LOWPASS

$$z^{-1} \longrightarrow \frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \quad \text{where } \alpha = \frac{\sin[(W_p - W_p')/2]}{\sin[(W_p + W_p')/2]}$$

W_p = Pass band frequency of LPF

W_p' = Passband frequency of New LPF

(b) LOWPASS TO HIGHPASS:

$$z^{-1} \longrightarrow -\left[\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}\right] \quad \text{where } \alpha = -\frac{\cos[(W_p' + W_p)/2]}{\cos[(W_p' - W_p)/2]}$$

W_p - Pass band frequency of LPF

W_p' - Pass band frequency of HPF

(c) LOWPASS TO BANDPASS:

$$z^{-1} \longrightarrow \frac{-\left[z^{-2} - \frac{2\alpha K}{1+K}\cdot z^{-1} + \frac{K-1}{K+1}\right]}{\frac{K-1}{K+1}\cdot z^{-2} - \frac{2\alpha K}{K+1} z^{-1} + 1}$$

where $\alpha = \frac{\cos[(W_u + W_l)/2]}{\cos[(W_u - W_l)/2]}$

and $K = \cot\left[\frac{W_u - W_l}{2}\right] \tan\frac{W_p}{2}$.

(d) LOWPASS TO BANDSTOP:

$$z^{-1} \longrightarrow \frac{z^{-2} - \frac{2\alpha}{1+K}\cdot z^{-1} + \frac{1-K}{1+K}}{\frac{1-K}{1+K}\cdot z^{-2} - \frac{2\alpha}{1+K} z^{-1} + 1}$$

where $\alpha = \frac{\cos[(W_u + W_l)/2]}{\cos[(W_u - W_l)/2]}$

and $K = \tan[(W_u - W_l)/2] \tan\frac{W_p}{2}$