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UNIT-II

DISCRETE FOURIER SERIES

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DISCRETE-FOURIER SERIES: Consider a sequence $x_p(n)$ with a Period of N Samples. So That $x_p(n) = x_p(n+lN)$. Since $x_p(n)$ is periodic, it can be represented as a weighted sum of Complex Exponentials whose frequency is integer Multiples of The fundamental frequency $\frac{2\pi}{N}$.

∴ These Periodic Complex Exponentials are of The form

$$\boxed{e^{j2\pi kn/N} = e^{j2\pi k(n+lN)/N}} \quad (1)$$

where l - an integer.

we know The DTFT of a sequence is Periodic it Consists of a Finite number of distinct - frequency Components. If number of frequency Components is N, Then The Harmonics are given by $\frac{2\pi k}{N}$, $k = 0, 1, 2 \cdots n-1$

∴ any periodic sequence can be represented as

$$\boxed{x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot e^{j2\pi kn/N}} \quad (2)$$

where $n = 0, \pm 1 \ldots$

→ and $X_p(k)$, $k = 0, 1, \ldots N-1$ are called DFS Coefficients.

To obtain Fourier Coefficients Multiply both-Sides of eq (2) by $e^{-j(2\pi/N)mn}$ and Sum The Product from $n=0$ to $n=N-1$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j(2\pi/N)mn} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} X_p(k) e^{j(2\pi/N)(k-m)n}$$

$$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi mn}{N}} = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \sum_{n=0}^{N-1} e^{j\frac{2\pi n}{N}(k-m)}$$

[∵ interchanging orders of Sum.]

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And $\sum_{n=0}^{N-1} e^{j(\frac{2\pi n}{N})(k-m)} = N$ if $k-m = 0, \pm N, \pm 2N \dots$
$= 0$ otherwise —— ③

$\Rightarrow \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}mn} = X_p(m)$ —— ④

By changing the index from 'm' to 'k' Fourier-Series Coefficient $X_p(K)$ in Eq ④ are obtained from $x_p(n)$ relation is

$$\boxed{X_p(K) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi Kn/N}} \quad \text{—— ⑤}$$

Eq ⑤ is Called DFS and Eq ① is called IDFS

By using both Equations we can observe That $x_p(n)$, $X_p(K)$ are Periodic with Period 'N' Samples.

i.e $DFS[x_p(n)] = X_p(K)$

PROPERTIES OF DFS :

(i) LINEARITY: Consider 2 Periodic Sequences $x_{1p}(n)$, $x_{2p}(n)$ with Period 'N'. Such that $DFS[x_{1p}(n)] = X_{1p}(K)$ and $DFS[x_{2p}(n)] = X_{2p}(K)$.

Then $DFS[a_1 x_{1p}(n) + a_2 x_{2p}(n)] = a_1 X_{1p}(K) + a_2 X_{2p}(K)$.

(ii) TIME-SHIFTING: If $x_p(n)$ is Periodic with Period 'N' Samples and $DFS[x_p(n)] = X_p(K)$

Then $DFS[x_p(n-m)] = e^{-j(2\pi/N)mK} X_p(K)$.

where $x_p(n-m)$ is a shifted version of $x_p(n)$.

(iii) PERIODIC CONVOLUTION: Let $x_{1p}(n)$ and $x_{2p}(n)$ be 2 Periodic Sequences with period N and $DFS[x_{1p}(n)] = X_{1p}(K)$, $DFS[x_{2p}(n)] = X_{2p}(K)$

If $X_{3p}(K) = X_{1p}(K) \cdot X_{2p}(K)$ Then Periodic-sequences $x_{3p}(n)$ with Fourier Series Coefficients $X_{3p}(K)$ is $x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) \cdot x_{2p}(n-m)$

i.e $DFS\left[\sum_{m=0}^{N-1} x_{1p}(m)\cdot x_{2p}(n-m)\right] = X_{1p}(K)\cdot X_{2p}(K).$

(IV) SYMMETRY PROPERTY :

→ If $DFS[x_p(n)] = X_p(K)$ Then $DFS[x_p^*(n)] = X_p^*(-K)$,

$DFS[x_p^*(-n)] = X_p^*(K)$ and $DFS\{Re[x_p(n)]\} = DFS\left[\frac{x_p(n)+x_p^*(n)}{2}\right]$

$$= \frac{1}{2}\left[X_p(K) + X_p^*(-K)\right]$$

$$= X_{pe}(K)$$

$$DFS\{jIm[x_p(n)]\} = DFS\left[\frac{x_p(n)-x_p^*(n)}{2}\right]$$

$$= \frac{1}{2}\left[X_p(K) - X_p^*(-K)\right]$$

$$= X_{po}(K)$$

→ we can write $x_p(n) = x_{pe}(n) + x_{po}(n)$ where

$$x_{pe}(n) = \frac{1}{2}\left[x_p(n) + x_p^*(-n)\right] \;\&$$

$$x_{po}(n) = \frac{1}{2}\left[x_p(n) - x_p^*(-n)\right]$$

Then $DFS[x_{pe}(n)] = DFS\left\{\frac{1}{2}[x_p(n) + x_p^*(-n)]\right\}$

$$= \frac{1}{2}\left[X_p(K) + X_p^*(K)\right]$$

$$= Re\{X_p(K)\}$$

$$DFS[x_{po}(n)] = DFS\left\{\frac{1}{2}[x_p(n) - x_p^*(-n)]\right\}$$

$$= \frac{1}{2}\left[X_p(K) - X_p^*(K)\right]$$

$$= jIm\{X_p(K)\}$$

→

DISCRETE FOURIER TRANSFORM :

We know $DTFT[x(n)] = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$.

$$IDTFT[X(e^{jw})] = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\cdot e^{jwn}\,dw.$$

There are Two drawbacks Associated with DTFT, i.e.

(i) Limits of summation which Extends b/w $-\infty$ to ∞ implying The length of the Signal must be infinitely long.

Note: DTFT is used for finite or infinite Sequences.
DFT is used for finite Sequences.

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(ii) The frequency variable 'ω' which is Continuous implying There is an infinite number of frequency points to be Computed.

* As a Result of first drawback DTFT Requires infinite time. To over come These Problems The limits of The summation are Reduced by Truncating an infinite signal to a finite Signal. This Process is called "windowing". because only a portion of The Actual discrete signal is available for Transform operation

* To overcome 2nd drawback the number of frequency points to be Computed is Restricted to a finite number. DFT is the Mathematical Procedure to Determine The harmonic or frequency content of a Discrete Signal.

The Fourier Transform of discrete-time signal is $X(\omega) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

When Fourier Transform is calculated at only Discrete points, it is Called Discrete Fourier Transform. It is denoted by $X(K)$. it is given as

DFT :- $$\boxed{X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}} \quad \text{——— (1)}$$

where K - sample of $X(\omega)$ in frequency Domain

we know Fourier Series Coefficient

$$\boxed{c_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}} \quad \text{——— (1a)}$$

Comparing these two Equations (1) & (1a) we get

$$\boxed{X(K) = N \cdot c_K}$$

The inverse DFT is given as

IDFT: $$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}}, \quad n = 0, 1, \cdots N-1 \quad (2)$$

~~we know Discrete Fourier Series~~

$$\cancel{x(n) = \sum_{k=0}^{N-1} c(k)\, e^{j2\pi kn/N}}$$

we know Discrete Fourier Series

$$x(n) = \sum_{k=0}^{N-1} c(k)\, e^{j2\pi kn/N} \quad (2a)$$

$$\left[\because c(k) = \frac{X(k)}{N}\right]$$

$$\therefore \boxed{\text{IDFT} = \frac{1}{N}\, x(n).}$$ from Eqs (2) and (2a)

Here 'N' is number of samples in $x(n)$. Hence DFT must be atleast N samples. If DFT[x(n)] has < N-point Then Aliasing will be takes place when IDFT is taken.

* Let us we defined a term $W_N = e^{-j\frac{2\pi}{N}}$ known as twiddle factor

Hence we can write DFT and IDFT Equations as follows.

DFT: $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}, \quad \text{where } k = 0, 1, \cdots N-1$$

IDFT: $$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn} \quad \text{where } n = 0, 1, \cdots N-1.$$

let The sequence $x(n)$ as a vector of 'N' samples. Then That sequence can be represented using Column matrix as

$$x_N(n) = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1} \quad \text{where } n = 0, 1 \cdots N-1$$

Similarly its DFT is represented as

$$X_N(k) = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1} \quad \text{where } k = 0, 1, \cdots N-1$$

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So the twiddle factors values can be represented as a matrix $[W_N]$ of size $N \times N$ as

$$[W_N] = \begin{bmatrix} W_N^{kn}\big|_{k=0,n=0} & W_N^{kn}\big|_{k=0,n=1} & \cdots & W_N^{kn}\big|_{k=0,n=N-1} \\ W_N^{kn}\big|_{k=1,n=0} & W_N^{kn}\big|_{k=1,n=1} & \cdots & W_N^{kn}\big|_{k=1,n=N-1} \\ \vdots & \vdots & \vdots & \vdots \\ W_N^{kn}\big|_{k=N-1,n=0} & W_N^{kn}\big|_{k=N-1,n=1} & \cdots & W_N^{kn}\big|_{k=N-1,n=N-1} \end{bmatrix}$$

$$= \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

i.e. $n \rightarrow$ row wise, K - column wise changes.

$\therefore$ The N-point DFT $[x(n)] = X(k)$ can be represented in Matrix form as $X_N(k) = [W_N]\, x_N(n)$

$IDFT(X_N(k)) = x_N(n)$ as $x_N(n) = \frac{1}{N}[W_N^*]\, X_N(k)$

where $W_N^* = W_N^{-kn}$

PERIODICITY OF W_N^{kn}:

Let $W_N^{kn} = e^{-j\frac{2\pi kn}{N}}$, we can calculate the values of W_N for $N=8$.

we know $W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_8 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$

Let $kn = a$ where $a = 0, 1, \ldots N-1$, Then

$kn = a =$	$W_8^a = e^{(-j\frac{2\pi}{8})a}$	Magnitude	phase
0	$W_8^0 = 1$	1	0
1	$W_8^1 = e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	1	$-\pi/4$
2	$W_8^2 = e^{-j\frac{\pi}{2}} = -j$	1	$-\pi/2$
3	$W_8^3 = e^{-j\frac{3\pi}{4}} = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$	1	$-\frac{3\pi}{4}$

contd..

$kn = a$	$W_8^a = e^{-j\pi/4\, a}$	Magnitude	phase
4	$W_8^4 = e^{-j\pi} = -1$	1	$-\pi$
5	$W_8^5 = e^{-j\frac{5\pi}{4}} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-\frac{5\pi}{4}$
6	$W_8^6 = e^{-j\frac{3\pi}{2}} = j$	1	$-\frac{3\pi}{2}$
7	$W_8^7 = e^{-j\frac{7\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$	1	$-\frac{7\pi}{4}$
8	$W_8^8 = e^{-j2\pi} = 1 = W_8^0$	1	$-2\pi = 0$
9	$W_8^9 = e^{-j\frac{9\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}} = W_8^1$	1	$\frac{-9\pi}{4} = \frac{-\pi}{4}$
10	$W_8^{10} = e^{-j\frac{5\pi}{2}} = -j = W_8^2$	1	$-\frac{5\pi}{2} = -\frac{\pi}{2}$
11	$W_8^{11} = e^{-j\frac{\pi}{4}\cdot 11} = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}} = W_8^3$	1	$-\frac{11\pi}{4} = -\frac{3\pi}{4}$
12	$W_8^{12} = e^{-j3\pi} = -1 = W_8^4$	1	$-3\pi = -\pi$
13	$W_8^{13} = e^{-j\frac{13\pi}{4}} = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = W_8^5$	1	$\frac{-13\pi}{4} = \frac{-5\pi}{4}$
14	$W_8^{14} = e^{-j\frac{7\pi}{2}} = j = W_8^6$	1	$-\frac{7\pi}{2} = \frac{-3\pi}{2}$
15	$W_8^{15} = e^{-j\frac{15\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}} = W_8^7$	1	$\frac{-15\pi}{4} = \frac{-7\pi}{4}$
16	$W_8^{16} = e^{-j4\pi} = 1 = W_8^8$	1	$-4\pi = -2\pi$

So we can write W_N^a is a periodic with period 'N'. That is $W_8^0 = W_8^8 = W_8^{16} = \ldots 1$

and $W_8^1 = W_8^9 = W_8^{17} = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

So In General we can write

$$\boxed{W_N^a = W_N^{a \pm N} = W_N^{a \pm 2N} \ldots}$$

This is known as <u>periodic Property</u> of TWIDDLE FACTOR.

and also we can write from the table

$W_8^1 = \frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$, $W_8^5 = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$

$W_8^2 = -j$ $\quad W_8^6 = +j$

In General we can write $\boxed{W_N^a = -W_N^{a \pm \frac{N}{2}}}$

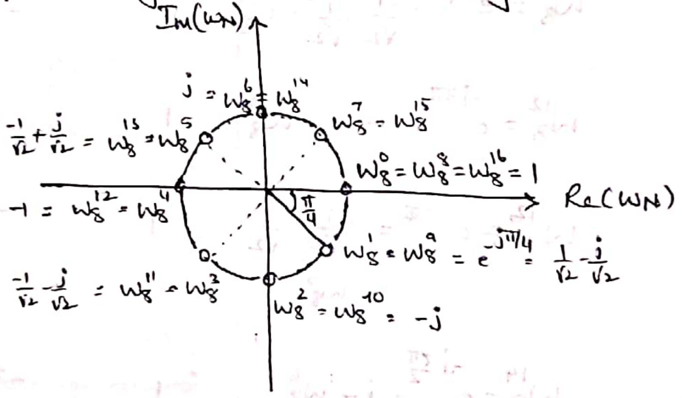
This is known as <u>symmetry property</u> of the Twiddle factor.

* Magnitude of Twiddle factor W_N^{nK} is $|W_N|$

$|W_N| = |e^{-j\frac{2\pi}{N}}| = |\cos\frac{2\pi}{N} - j\sin\frac{2\pi}{N}|$ and

* phase Angle of Twiddle factor is $\angle e^{-j\frac{2\pi}{N}} = -\frac{2\pi}{N}$.

The symmetry and periodicity can be shown as



<u>ZERO PADDING</u>:-

let a signal with length 'L' is $x(n) = \{x(0), x(1) \ldots x(L-1)\}$ i.e it has L samples, minimum number of Equally-Spaced frequency points can be calculated b/w $0 \to 2\pi$ without time domain aliasing, so frequency (determine) Resolution is $\frac{2\pi}{L}$. This frequency Resolution is said to be poor if straight line interpolation (inserting into a block) of frequency samples are different from Actual magnitude and phase Response. If we increase number of points b/w $0 \to 2\pi$ we can get better Representation of Magnitude and phase Response i.e The frequency Resolution is Improved by Adding zero's to signal as. $x(n)=\{x(0),x(1)\cdots x(L-1),0,0\cdots 0\}$

* In order to have 'N' frequency points b/w 0 and 2π, we add N-L zeros to the sequence. Then the sequence length is = to 'N'. Then The frequency Resolution of N-point DFT is $\frac{2\pi}{N}$.

Example problems:

1. Find the DFT of sequence $x(n)=\{1,1,0,0\}$ and IDFT of $Y(K)=\{1,0,1,0\}$

Sol: let $N=L=4$

we know $X(K)=\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi Kn}{N}}$, $K=0,1\cdots N-1$

Then $X(0)=\sum_{n=0}^{3} x(n) = x(0)+x(1)+x(2)+x(3)$
$= 1+1+0+0$
$= 2$

$X(1)=\sum_{n=0}^{3} x(n)\cdot e^{-j\frac{2\pi n}{4}} = x(0)+x(1)e^{-j\pi/2}+x(2)e^{-j\pi}+x(3)e^{-3j\pi/2}$
$= 1+\cos\frac{\pi}{2}-j\sin\frac{\pi}{2}$
$= 1-j$

$X(2)=\sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$
$= 1+\cos\pi-j\sin\pi$
$= 0$

$X(3)=\sum_{n=0}^{3} x(n) e^{-j\frac{3n\pi}{2}} = x(0)+x(1)e^{-j\frac{3\pi}{2}}+x(2)e^{-j3\pi}+x(3)e^{-j\frac{9\pi}{2}}$
$= 1+\cos\frac{3\pi}{2}-j\sin\frac{3\pi}{2}$
$= 1+j$

$\therefore$ DFT $\{x(n)\} = X(K) = \{2, 1-j, 0, 1+j\}$

we know IDFT$[Y(K)] = y(n) = \frac{1}{N}\sum_{K=0}^{N-1} Y(K)\cdot e^{\frac{j2\pi Kn}{N}}$.

$\Rightarrow y(0) = \frac{1}{4}\sum_{K=0}^{3} Y(K)$, $n=0,1,2,3$
$= \frac{1}{4}\{Y(0)+Y(1)+Y(2)+Y(3)\}$
$= \frac{1}{4}(1+0+1+0) = 0.5$

$$y(1) = \frac{1}{N}\sum_{K=0}^{3} Y(K)\cdot e^{j\pi K/2}$$

$$= \frac{1}{4}\left[Y(0)+Y(1)e^{j\pi/2} + Y(2)e^{j\pi} + Y(3)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1+0+\cos\pi + j\sin\pi + 0\right]$$

$$= \frac{1}{4}\left[1+0-1+0\right] = 0$$

$$y(2) = \frac{1}{4}\sum_{K=0}^{3} Y(K).e^{j\pi K}$$

$$= \frac{1}{4}\left\{Y(0)+Y(1)e^{j\pi} + Y(2)e^{j2\pi} + Y(3)e^{j3\pi}\right\}$$

$$= \frac{1}{4}\left\{1+0+\cos\pi + j\sin 2\pi + 0\right\}$$

$$= \frac{1}{4}\left\{1+0+1+0\right\} = 0.5$$

$$y(3) = \frac{1}{4}\sum_{K=0}^{3} Y(K)e^{j3\pi K/2}$$

$$= \frac{1}{4}\left\{Y(0)+Y(1)e^{j\frac{3\pi}{2}} + Y(2)e^{j3\pi} + Y(3)e^{j\frac{9\pi}{2}}\right\}$$

$$= \frac{1}{4}\left\{1+0+\cos 3\pi\, j\sin 3\pi + 0\right\}$$

$$= \frac{1}{4}\left\{1+0+(-1)+0\right\} = 0$$

$\therefore\ y(n) = \{0.5, 0, 0.5, 0\}$

2) Find DFT of a sequence $x(n)=1$ for $0 \le n \le 2$
$= 0$ otherwise

for (i) $N=4$ (ii) $N=8$ plot $|X(K)|$ & $\angle X(K)$.

Sol: given length of sequence $L=3$ $(0 \le n \le 2)$

But $N=4$ so we can add a zero to given sequence

$N-L$ zeros $= 4-3=1$

$\therefore$ sequence is $x(n) = \{1, 1, 1, 0\}$

we know $DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi kn}{N}}, K=0,\cdots N-1$

$$\therefore X(0) = \sum_{n=0}^{3} x(n)$$

$$= x(0)+x(1)+x(2)+x(3)$$

$$= 1+1+1+0 = 3$$

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$\therefore$ Magnitude $|X(0)| = 3$

phase $\angle X(0) = 0$

for $K=1$, $X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$

$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$

$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + 0$

$= 1 - j - 1 = -j$

$\therefore$ Magnitude $|X(1)| = 1$

phase $\angle X(1) = -\pi/2$

$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$

$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$

$= 1 - 1 + 1 = 1$

$\therefore$ Magnitude $|X(2)| = 1$

phase $\angle X(2) = -\pi = 0$

$X(3) = \sum_{n=0}^{3} x(n) e^{-j3n\pi/2}$

$= x(0) + x(1)e^{-3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$

$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$

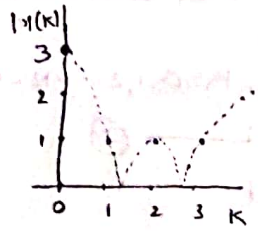
$= 1 + j - 1 = j$

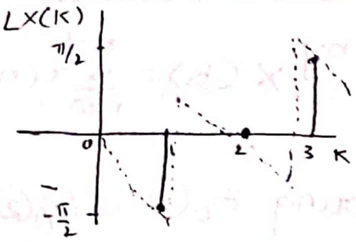
$\therefore$ Magnitude $|X(3)| = 1$

phase $\angle X(3) = -\frac{3\pi}{2} = \frac{\pi}{2}$

$\therefore X(K) = \{3, -j, 1, j\}$

Plots of $|X(K)|$, $\angle X(K)$





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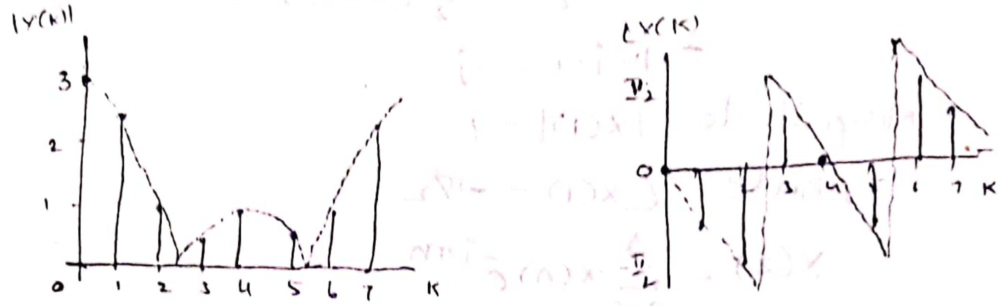
Similarly for N=8, we get

$X(K) = \{3, 1.707 - j1.707, -j, 0.293 + j0.293, 1, 0.293 - j0.293, j, 1.707 + j1.707\}$

$|X(K)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$

$\angle X(K) = \{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0, -\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\}$



3. Determine 8-point DFT of sequence $x(n)$, where $x(n) = \{1,1,1,1,1,1,0,0\}$

Sol: $X(K) = \{6, -0.707 - j1.707, 1 - j, 0.707 + j0.293, 0, 0.707 - j0.293, 1 + j, -0.707 + j1.707\}$

4. Find IDFT of $X(K) = \{4, 1, 0, 1-j, 1, 0, 1+j, 1\}$

Sol: $x(n) = \{ \qquad \}$

RELATION BETWEEN DFT AND OTHER TRANSFORMS:

(i) RELATION TO THE FOURIER TRANSFORM:

The Fourier Transform of a finite duration sequence $x(n)$ having length N is given by

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \quad \text{——— (1)}$$

The Discrete Fourier Transform of a $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}, \quad K = 0,1,2 \cdots N-1 \quad \text{——— (2)}$$

Comparing Eq(1) & Eq(2) we get

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{N}}, \quad K = 0, 1 \ldots N-1$$

(ii) RELATION TO THE Z-TRANSFORM

Let the Z-Transform of a sequence having finite duration 'N' as $X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$ ——(1), and

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi K n}{N}} \quad \text{——(2)}$$

Comparing Eq(1) & Eq(2) we get

$$X(K) = X(e^{j\omega})\Big|_{z = e^{+j\frac{2\pi K}{N}}} , \quad K = 0,1,2 \ldots N-1$$

PROPERTIES OF DISCRETE FOURIER TRANSFORM :-

(i) PERIODICITY :-

If $X(K)$ is N-point DFT of a finite duration sequence $x(n)$ then $x(n+N) = x(n)$ for all 'n'

$$X(K+N) = X(K) \text{ for all 'K'}.$$

(ii) LINEARITY :-

If two finite duration sequences $x_1(n)$ & $x_2(n)$ are linearly combined as $x_3(n) = a x_1(n) + b x_2(n)$ then $DFT[x_3(n)] = X_3(K) = aX_1(K) + bX_2(K)$.

If $x_1(n)$ has length N_1 and $x_2(n)$ has length N_2 then maximum length of $x_3(n)$ will be $N_3 = Max(N_1, N_2)$

Let N_1-point DFT is $X_1(K) = \sum_{n=0}^{N_1-1} x_1(n) e^{-j\frac{2\pi K n}{N_1}}$

where $0 \le K \le N_1-1$

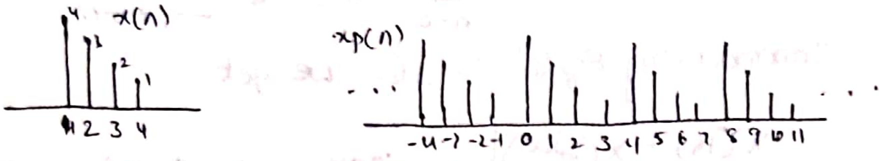
Let N_2-point DFT is $X_2(K) = \sum_{n=0}^{N_2-1} x_2(n) e^{-j\frac{2\pi K n}{N_2}}$

where $0 \le K \le N_2-1$

$\therefore DFT[a x_1(n) + b x_2(n)] = aX_1(K) + bX_2(K)$.

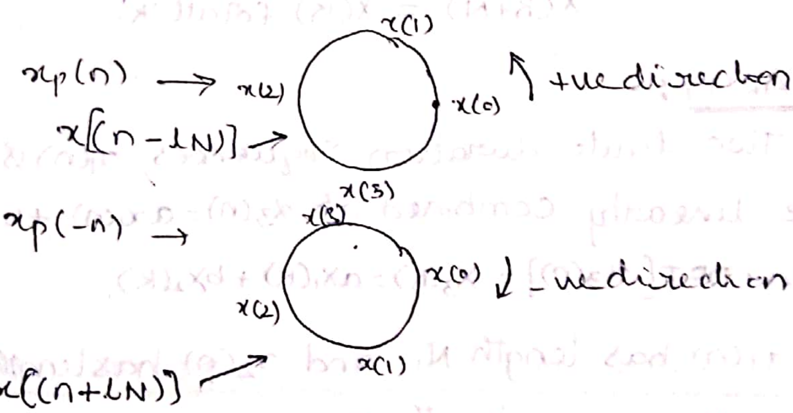
(iii) circular shift of a sequence.

The Periodic Extention of a sequence can be represented as $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$



If we consider periodic sequence $x_p(n)$ is wrapping the finite duration sequence $x(n)$. This can be expressed using a circle. representation will be of Modulo-N

$$x_p(n) = x[(n \text{ modulo } N)]$$



In General shifted version $x(n)$ can be shown as

$$x(n) = [x(0), x(1), x(2) \dots, x(N-2), x(N-1)]$$

$$x((n-1))_N = [x(N-1), x(0), x(1) \dots, x(N-3), x(N-2)]$$

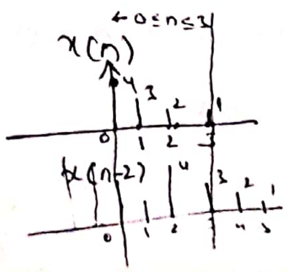
$$x((n-2))_N = [x(N-2), x(N-1), x(0), \dots x(N-4), x(N-3)]$$

$$\vdots$$

$$x((n-K))_N = [x(N-K), x(N-K+1), x(N-K+2) \dots x(N-K-1)]$$

$$x((n-N))_N = [x(0), x(1), x(2) \dots x(N-2), x(N-1)]$$

ex:



i.e $x(n) = x((n-N))_N$

$\Rightarrow x(n-N+N) = x((n-N))_N$

Similarly $x((n-m))_N = x(N+n-m)$

Now. Come to the property

$DFT[x(n)] = X(K)$ Then $DFT[x((n-m))_N] = e^{-j\frac{2\pi km}{N}} \cdot X(K)$.

proof: Apply above concept on following the below.

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{m-1} x((n-m))_N \cdot e^{-j\frac{2\pi kn}{N}} + \sum_{n=m}^{N-1} x((n-m))_N e^{-j\frac{2\pi kn}{N}}$$

But we know $x((n-m))_N = x(N+n-m)$

$$\therefore \text{let } \sum_{n=0}^{m-1} x((n-m))_N e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{m-1} x(N+n-m) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{l=N-m}^{N-1} x(l) e^{-j\frac{2\pi K(l+m-N)}{N}}$$

let $l = N+n-m$

$$= \sum_{l=N-m}^{N-1} x(l) e^{-j\frac{2\pi k(l+m)}{N}} \quad \text{--- (1)}$$

$\Rightarrow n = l+m-N$.

$[\because e^{j2\pi k} = 1 \text{ for } k = 0, 1, 2 \ldots]$

Similarly

$$\text{let } \sum_{n=m}^{N-1} x((n-m))_N e^{-j\frac{2\pi kn}{N}} = \sum_{n=m}^{N-1} x(n-m+N) e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{l=0}^{N-m-1} x(l) \cdot e^{-j\frac{2\pi k(l+m)}{N}} \quad \text{--- (2)}$$

let $l = n-m+N$

$n = l+m-N$

from Eq (1) & Eq (2) we can write

$$\sum_{n=0}^{N-1} x((n-m))_N e^{-j\frac{2\pi kn}{N}} = \sum_{l=N-m}^{N-1} x(l) e^{-j\frac{2\pi k(l+m)}{N}} + \sum_{l=0}^{N-m-1} x(l) e^{-j\frac{2\pi k(l+m)}{N}}$$

$$= \sum_{l=0}^{N-1} x(l) e^{-j\frac{2\pi k(l+m)}{N}}$$

$$= e^{-j\frac{2\pi km}{N}} \cdot X(K)$$

(iv) TIME REVERSAL OF THE SEQUENCE

The time Reversal of an N-point sequence $x(n)$ is obtained by wrapping the sequence $x(n)$ around the circle in clockwise direction. It is denoted as $x((-n))_N$.

i.e $\quad x((-n))_N = x(N-n) \qquad 0 \le n \le N-1.$

If $DFT[x(n)] = X(K)$ Then $DFT[x((-n))_N] = DFT[x(N-n)]$

$$= X((-K))_N = X(N-K).$$

[changing the index from ~~x~~ to N−n = ~~m~~ we get

$$DFT[x(\cancel{N-m})] = \sum_{m=0}^{N-1} \cancel{x(m)} e^{-j\cancel{2\pi k(N-m)/N}}$$

$$= \cancel{\sum_{m=0}^{N-1} x(m)} \cdot e^{\cancel{j2\pi km/N}}$$

$$= \sum_{m=0}^{N-1} \cancel{x(m)} e^{-j\cancel{2\pi km(N-k)/N}} \qquad [\because \cancel{e^{j2\pi k} = 1} \text{ for } k=0,1,2.$$

$$= \cancel{X(N-K)}.]$$

Proof:

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) e^{-j\frac{2\pi K n}{N}} \qquad [\ e^{j2\pi k} \quad N-n=a$$

$$n = N-a$$

$$= \sum_{n=0}^{N-1} x(a) e^{-j\frac{2\pi KN}{N}} \cdot e^{+j\frac{2\pi ka}{N}} \qquad [\because e^{j2\pi k} = 1 \text{ for } k=\ldots$$

$$= \sum_{n=0}^{N-1} x(a) e^{+j\frac{2\pi ka}{N}}$$

$$= X((-K))_N = X(N-K)$$

(v) CIRCULAR FREQUENCY SHIFT :

If $DFT[x(n)] = X(K)$ Then $DFT[x(n)e^{j2\pi l n/N}] = X((K-l))_N$

Proof:

$$DFT\left[x(n)e^{j\frac{2\pi l n}{N}}\right] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k n}{N}} \cdot e^{j\frac{2\pi l n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n(K-l)/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi n\frac{(N+K-l)}{N}}$$

$$= X(N+K-l) = X((K-l))_N$$

vi) COMPLEX CONJUGATE PROPERTY:

If $DFT[x(n)] = X(K)$ Then $DFT[x^*(n)] = X^*(N-K) = X^*((-K))_N$

Proof: $$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) \cdot e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) e^{-j2\pi n(N-K)/N}\right]^*$$

$$= X^*(N-K) = X^*((-K))_N$$

Similarly $$IDFT[X^*(K)] = \left(\frac{1}{N}\sum_{K=0}^{N-1} X^*(K) e^{j2\pi kn/N}\right)^*$$

$$= \frac{1}{N}\left[\sum_{K=0}^{N-1} X(K) e^{-j2\pi kn/N}\right]^*$$

$$= \frac{1}{N}\left[\sum_{K=0}^{N-1} X(K) \cdot e^{j2\pi K(N-n)/N}\right]^*$$

$$= x^*(N-n).$$

(vii) CIRCULAR CONVOLUTION :-

let Two finite Sequences $x_1(n), x_2(n)$ of Length 'N' with DFTs as $X_1(K), X_2(K)$ Respt.

So we have a sequence $x_3(n)$ with DFT is $X_3(K)$

where $X_3(K) = X_1(K) \cdot X_2(K)$.

Proof:

let The Convolution of Two sequences is

$$x_{3p}(n) = \sum_{m=0}^{N-1} x_{1p}(m) \cdot x_{2p}(n-m).$$

$$x_3((n))_N = \sum_{m=0}^{N-1} x_{1p}((m))_N \cdot x_2((n-m))_N$$

But for $0 \le n \le N-1$, $x_3((n))_N = x_3(n)$, $x_1((m))_N = x_1(m)$

$$\therefore x_3(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2((n-m))_N \quad \text{———} (1)$$

Eq (1) represents the circular Convolution of $x_1(n), x_2(n)$

$$x_3(n) = x_1(n) \, (N) \, x_2(n)$$

$$\therefore DFT[x_1(n) \, (N) \, x_2(n)] = X_1(K) \cdot X_2(K)$$

(viii) CIRCULAR CORRELATION:

for complex valued sequences $x(n), y(n)$ if

$DFT[x(n)] = X(K)$, $DFT[y(n)] = Y(K)$ Then

$$DFT[\tilde{r}_{xy}(l)] = DFT\left[\sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N\right]$$

$$= X(K) \cdot Y^*(K)$$

(ix) MULTIPLICATION OF TWO SEQUENCES:

If $DFT[x_1(n)] = X_1(K)$, $DFT[x_2(n)] = X_2(K)$ ~~Then~~ Then

$$DFT[x_1(n) \cdot x_2(n)] = \frac{1}{N}[X_1(K) \, \text{Ⓝ} \, X_2(K)]$$

Proof:

$$DFT[x_1(n) x_2(n)] = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(n) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{N} \cdot \sum_{l=0}^{N-1} X_1(l) \cdot e^{+j\frac{2\pi ln}{N}} \cdot x_2(n) e^{-j\frac{2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j\frac{2\pi n}{N}(k-l)}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{-j\frac{2\pi n}{N}(N+k-l)}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot X_2((k-l))_N$$

$$= \frac{1}{N}[X_1(K) \, \text{Ⓝ} \, X_2(K)]$$

(x) PARSEVAL'S THEOREM:-

If $DFT[x(n)] = X(K)$ and $DFT[y(n)] = Y(K)$ Then

$$\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot Y^*(K)$$

let $$\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \sum_{n=0}^{N-1} x(n) \left[\frac{1}{N} \sum_{K=0}^{N-1} Y(K) \cdot e^{j\frac{2\pi kn}{N}}\right]^*$$

$$= \sum_{K=0}^{N-1} \frac{1}{N} Y^*(K) \cdot \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{K=0}^{N-1} Y^*(K) \cdot X(K).$$

COMPARISION BETWEEN CIRCULAR & LINEAR CONVOLUTION

The linear convolution of sequences $x(n)$ and $h(n)$ produces a Result $y(n)$ which contains number of samples as $N = N_1 + N_2 - 1$ where N_1 is length of $x(n)$, N_2 is length of $h(n)$.

But In circular convolution, After the convolution the sequence having number of samples as $N = \text{Max}(N_1, N_2)$. If $N_2 < N_1$ we must add $N_1 - N_2$ number of zeros to sequence $h(n)$. so both sequences periodic with N.

linear convolution can be used to find the Response of a filter. Circular convolution can't be used to find Response of a linear filter without zero padding.

We have 2 Methods to find circular convolution of Two sequences. They are

(1) Concentric Circle Method
(2) Matrix Multiplication Method.

CONCENTRIC CIRCLE METHOD:-

Let Two sequences $x_1(n)$ and $x_2(n)$ The circular convolution of these Two sequences is $x_3(n) = x_1(n) \; N \; x_2(n)$ can be found by using following steps.

1. Graph N samples of $x_1(n)$ as Equally spaced Points around an outer circle in Anticlock-wise direction

2. start at the same point as $x_1(n)$ graph N samples of $x_2(n)$ as Equally spaced points around an inner circle in clockwise direction.

3. Multiply corresponding samples on the Two circles and sum the products to produce output.

4. Rotate The inner circle one sample at a time in Anticlock wise direction and go to step 3.

5. Repeat step 4 until the inner circle first Sample lines up with the first Sample of the Exterior circle once again.

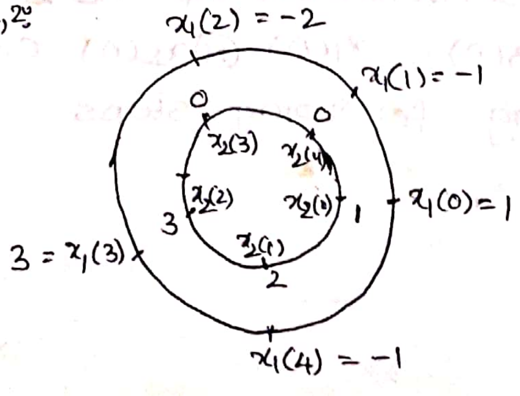
Ex: find circular convolution of Two sequences $x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3\}$.

Sol: To find circular convolution both seq Must be of Same length. so we add Two zeros to $x_2(n)$.

$\therefore x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3, 0, 0\}$

Step 1, 2:



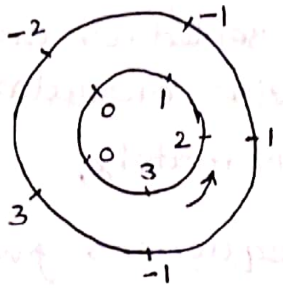
Step 3:

$$y(0) = x_1(0)\cdot x_2(0) + x_1(1)x_2(4) + x_1(2)x_2(3) + x_1(3)x_2(2) + x_1(4)\cdot x_2(1).$$

$$= 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1)$$

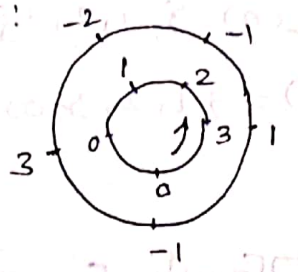
$$= 8$$

Step 4:

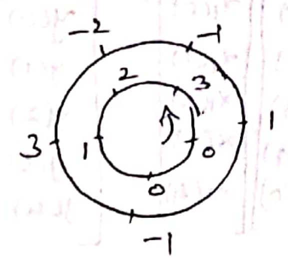


$$y(1) = 1(2) + 1(-1) + 0(-2) + 0(3) + 3(-1)$$
$$= -2$$

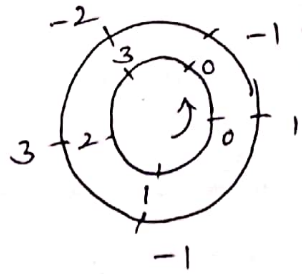
Step 5:



$$y(2) = 1(3) - 1(2) - 2(1) + 3(0) + (-1)(0)$$
$$= -1$$



$$y(3) = 1(0) + (-1)(3) - 2(2) + 3(1) - 1(0)$$
$$= -4$$



$$y(4) = 1(0) - 1(0) - 2(3) + 3(2) - 1(1)$$
$$= -1$$

$$\therefore \quad y(n) = \{8, -2, -1, -4, -1\}$$

MATRIX MULTIPLICATION METHOD :- In This Method circular convolution of Two Sequences $x_1(n)$, $x_2(n)$ can be obtained by representing The Sequences in Matrix form as follows

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(1) & x_2(0) \end{bmatrix}_{N\times N} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix}_{N\times 1} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}_{N\times 1}$$

The sequence $x_1(n)$ is represented as column-matrix. $x_2(n)$ repeated via circular shift and is represented by $N\times N$ matrix.

Ex: find convolution of two sequences given by

$x_1(n) = \{1, -1, -2, 3, -1\}$, $x_2(n) = \{1, 2, 3\}$

by adding zeros to $x_2(n) = \{1, 2, 3, 0, 0\}$

given $N=5$ so matrix is

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}}_{x_2(n)} \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix}}_{x_1(n)} = \underbrace{\begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}}_{y(n)}$$

$\therefore\ y(n) = \{8, -2, -1, -4, -1\}$

find convolution of above using DFT & IDFT

i.e. $X_3(K) = X_1(K)X_2(K)$

$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi Kn}{N}}$ $\longrightarrow$ find

$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi Kn}{N}}$ $\longrightarrow$ find

Next $X_3(K) = X_1(K)\cdot X_2(K)$

Then $x_3(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_3(K)\, e^{\frac{j2\pi Kn}{N}}$

~~LINEAR-C~~

LINEAR CONVOLUTION FROM CIRCULAR CONVOLUTION (or)

In signal processing applications we - interested in linear convolution of Two sequ-unces. let Two finite duration sequences $x(n)$ and $h(n)$ with samples N_1 and N_2 resp.

The linear convolution $y(n) = x(n) * h(n)$

$$\Rightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=0}^{N_1+N_2-1} x(k) \cdot h(n-k)$$

we have here a finite sequence.

$y(n)$ has length of N_1+N_2-1 samples.

But in circular convolution samples for $y(n)$ are $N = \max(N_1, N_2)$

If $N_2 < N_1$, to find circular convolution we must add zero's ~~to~~ $N_1 - N_2$ to $h(n)$ sequence.

So To find circular convolution $x(n), h(n)$ must have N_1+N_2-1 samples

So we must add N_2-1 zeros to $x(n)$ and N_1-1 zeros to $h(n)$.

LINER CONVOLUTION USING DFT:

To perform linear convolution using DFT it is necessary to Evaluate DFT at sufficiently large number of points so that we can avoid occuring of time-aliasing Effect.

let $x_1(n), h(n)$ of lengths N_1 and N_2 Then $y_3(n) = x_1(n) * h(n)$ is obtained by $Y_3(K) = X_1(K) \cdot H(K)$ and taking IDFT of $Y_3(K)$.

N

Ex: perform linear convolution using DFT

where $x(n) = \begin{cases} 1 & \text{for } n=0 \\ 0.5 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$

$h(n) = \begin{cases} 0.5 & \text{for } n=0 \\ 1 & \text{for } n=1 \\ 0 & \text{otherwise} \end{cases}$

Sol: first find $N = N_1 + N_2 - 1$

Then find $X(K)$ and $H(K)$

Multiply both $X(K)$ and $H(K)$ to $Y(K)$

$$Y(K) = X(K) \cdot H(K)$$

then find $y(n) = IDFT[Y(K)]$.

Ex: given 8-pt DFT of a sequence $x(n) = 1 \quad 0 \le n \le 3$
$= 0 \quad 4 \le n \le 7$

then Compute DFT of (i) $x_1(n) = \begin{cases} 1 & n=0 \\ 0 & 1 \le n \le 4 \\ 1 & 5 \le n \le 7 \end{cases}$

(ii) $x_2(n) = \begin{cases} 0 & 0 \le n \le 1 \\ 1 & 2 \le n \le 5 \\ 0 & 6 \le n \le 7 \end{cases}$

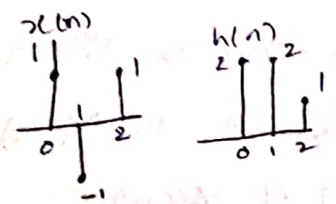
first find $X(K)$

then to find $X_1(K)$ apply Time shifting

$$x_1(n) = x((n-(-3)))_N \Rightarrow X_1(K) = DFT[x((n-m))_N]$$

$$\xrightarrow{DFT} [x((n-m))_N] = X(K) \cdot e^{\frac{-j2\pi Km}{N}}$$

→ use DFT to compute linear convolution of signals shown



FILTERING LONG DURATION SEQUENCES: Let aninput sequence $x(n)$ is of long duration is to be - Processed with a system having impulse - Response of finite duration. by Convolving Two sequences. Because of length of input sequence it would not be practical to store it all before Performing linear Convolution. So The input - Sequence must be divided up into blocks. we have Two methods for This sectioned convolution. They are

(i) Overlap- Save Method

(ii) Overlap- Add Method.

(i) overlap - save Method :-

Let input sequence of length N_1, and an impulse sequence of length N_2. So In This input Sequence is divided into blocks of Size $N = N_1+N_2-1$, for Each block consists of Last(N_2-1) data points of previous block followed by N_1 New data points. Now impulse sequence length is increased with adding N_1-1 zeros. then $N = N_1+N_2-1$, and $y_k(n) = x_k(n) \circledN h(n)$.

Ex: find $y(n)$ with $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ and $h(n) = \{1,1,1\}$

Sol: Let $N_1 = 10$, $N_2 = 3$ $\qquad N = N_1+N_2-1 = 12$

given input sequence can be divided as follows

$x_1(n) = \{\underbrace{0,0}_{M-1}, \underbrace{3,-1,0}_{M=3 \text{ data pt}}\}$

$x_2(n) = \{-1,0, 1,3,2\}$

$x_3(n) = \{3,2, 0,1,2\}$

$x_4(n) = \{1,2, 1,0,0\}$

here Each input Sequence length $N_1 = 5$

$h(n)$ length $N_2 = 3$

But $N = max(N_1, N_2)$

$= 5$, 2 zeros added to $h(n)$

$\therefore h(n) = \{1, 1, 1, 0, 0\}$

$\therefore y_1(n) = x_1(n) \circledN h(n) = \{\underbrace{-1, 0}, 3, 2, 2\}$

$y_2(n) = x_2(n) \circledN h(n) = \{\underbrace{4, 1}, 0, 4, 6\}$

$y_3(n) = x_2(n) \circledN h(n) = \{\underbrace{6, 7}, 5, 3, 3\}$

$y_4(n) = x_4(n) \circledN h(n) = \{\underbrace{1, 3}, 4, 3, 1\}$

discard

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$ ✓

(ii) Overlap Add Method:

let a block contains 3 data bits.

Then $x_1(n) = \{3, -1, 0, 0, 0\}$

$x_2(n) = \{1, 3, 2, 0, 0\}$

$x_3(n) = \{0, 1, 2, 0, 0\}$

$x_4(n) = \{1, 0, 0, 0, 0\}$

h(n) x(n) y(n)

Then $y_1(n) = x_1(n) \circledN h(n) = \{3, 2, 2, -1, 0\}$

$y_2(n) = x_2(n) \circledN h(n) = \{1, 4, 6, 5, 2\}$

$y_3(n) = x_3(n) \circledN h(n) = \{0, 1, 3, 3, 2\}$

$y_4(n) = x_4(n) \circledN h(n) = \{1, 1, 1, 0, 0\}$

3, 2, 2, -1, 0

↓ ↓ add

1, 4, 6, 5, 2

↓ ↓ add

0, 1, 3, 3, 2

↓ ↓

1, 1, 1, 0, 0

neglect

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$ ✓

	1	1	1
3	3	3	3
-1	-1	-1	-1
0	0	0	0

3, 2, 2, -1, 0

	0	0	1	1	1
-1	0	0	-1	-1	-1
0	0	0	0	0	0
1	0	0	1	1	1
3	0	0	3	3	3
2	0	0	2	2	2

0, 0, -1, -1, 0, 4, 6, 5, 2

0 0 3, -1, 0 1 1 1 0 0

	0	0	1	1	1
0	0	0	0	0	0
0	0	0	0	0	0
3	0	0	3	3	3
-1	0	0	-1	-1	-1
0	0	0	0	0	0

0 0 3, 2, 2, -1, 0

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 6 \end{bmatrix}$$

-1 + 2

-1

Radha Swami

UNIT-III

FAST FOURIER TRANSFORM

Dt: 03-02-09.

INTRODUCTION: In Previous chapter we have seen the use of DFT to find Response of linear filter. By doing direct computation of DFT we have one Disadvantage. i.e It requires more number of Computations (Multiplications and Additions). To overcome This Disadvantage we have a set of Algorithms known as Fast Fourier Transform (FFT) Proposed by Cooley and Tukey in 1965

The Main Advantage of FFT is Reducing The Computation time Required to compute DFT. The basic operation of FFT is Decomposition & Breaking The Transform into smaller Transform And combining Them to get total Transform.

WHAT is FFT?

The FFT is an algorithm used to compute The DFT. It uses The Symmetry and periodicity properties of Twiddle factor W_N^K to effectively Reduce The DFT Computation time. It is based on fundamental principle of Decomposing The Computation of DFT of a sequence of length 'N' into successively smaller DFTs. It provides Speed improvement factor.

NEED FOR FFT:

The DFT of a Sequence can be Directly Evaluated using The formula X(K) where

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j 2\pi \frac{kn}{N}} \quad 0 \le K \le N-1 \quad \text{—(1)}$$

The Equation ① in Twiddle factor Notation is given by $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nK}$, $0 \le K \le N-1$ ——②

where $W_N = e^{-j2\pi/N}$.

from Eq ②

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{nK}, \quad 0 \le K \le N-1$$

$$= \sum_{n=0}^{N-1} \{Re[x(n)] + j\,Im[x(n)]\}\{Re[W_N^{nK}] + j\,Im[W_N^{nK}]\} \quad \text{——③}$$

$$= \sum_{n=0}^{N-1} Re[x(n)] \cdot Re[W_N^{nK}] - \sum_{n=0}^{N-1} Im[x(n)]\, Im[W_N^{nK}]$$

$$+ j\left\{\sum_{n=0}^{N-1} Im[x(n)] Re[W_N^{nK}] + \sum_{n=0}^{N-1} Re(x(n)) \cdot Im(W_N^{kn})\right\} \quad \text{——④}$$

from Eq ③ we can observe That, to Compute one value of X(K). number of Complex Multi-plications Required is 'N'. for Total 'N' - values of X(K) we Require N^2 Complex. Multiplications.

Similarly for Evaluation of one value of X(K) we require (N-1) Complex Additions. So for total 'N' number of values we require - N(N-1) Complex Additions.

from Eq ④ we can write number of Real Multiplications are $4N^2$ and Real Additions are $N[4(N-1)+2]$ real Additions (to combine sum to get real & imaginary part). i.e $(4N-2)N$.

i.e The Direct Evaluation of DFT require more number of Computations.

It is Basically inefficient because it does'nt use Symmetry [$W_N^K = -W_N^{K+\frac{N}{2}}$] and periodicity [$W_N^K = W_N^{K+N}$] properties. of Twiddle factor.

FFT uses These basic properties, and Reduces number of Complex Multiplications to Perform DFT from N^2 to $\frac{N}{2}\log_2 N$.

Ex: If N=16, multiplications for DFT Directly are $N^2 = 256$, But by The use of FFT we need

$$\frac{N}{2}\log_2 N = \frac{16}{2}\log_2 2^4 = 32.$$

Speed improvment factor $= \frac{256}{32} = 8$

TYPES OF FFT ALGORITHMS:

Basically There are Two classes of FFT Algorithms. They are

1. Decimation-in time
2. Decimation-in frequency

Decimation-in time, In This, The Sequence for which we need DFT is successively divided into smaller sequences and DFT of These subsequences are Combined in a Certain Pattern to obtain The Required DFT of Entire Sequence.

Decimation-in frequency, The frequency Samples of DFT are decomposed into Smaller and Smaller Subsequences in a similar Manner.

Decomposition – Separate into Small Elements

Decimation — Remove from a large (a 1/10th part)

DECIMATION-IN-TIME ALGORITHM:

This Algorithm is also known as Radix-2 DIT-FFT algorithm which Means Number of Output points 'N' can be Expressed as a Power of 2, i.e $N = 2^M$, where M - integer.

Let $x(n)$ is an N-point sequence, Decimate or Break This point sequence into Two Sequences of length $\frac{N}{2}$, i.e as a Combination of Even and Odd.

$$\therefore \quad x_e(n) = x(2n), \quad n = 0, 1, 2 \dots, \frac{N}{2} - 1. \quad \text{——— (a)}$$

$$x_o(n) = x(2n+1), \quad n = 0, 1, 2 \dots, \frac{N}{2} - 1. \quad \text{——— (b)}$$

we know N-point DFT. $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad K = 0, 1 \dots N-1$

$$\Rightarrow X(K) = \sum_{\substack{n=0 \\ (\text{Even})}}^{N-1} x(n) W_N^{kn} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n) W_N^{nk}. \quad \text{——— (1)}$$

from Eq (a) & Eq (b) & Eq (1)

$$\Rightarrow X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nK} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot W_N^{(2n+1)K}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nK} + W_N^K \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk} + W_N^K \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_N^{2nk}. \quad \text{——— (2)}$$

we have $W_N = e^{-j2\pi/N}$. $\Rightarrow W_N^2 = \left[e^{-j\frac{2\pi}{N}}\right]^2 = e^{-j\frac{2\pi}{N/2}} = W_{\frac{N}{2}}$.

$$\therefore \quad W_N^2 = W_{N/2} \quad \text{——— (3)}$$

from Eq(2) & Eq(3)

$$X(K) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{\frac{N}{2}}^{nk}}_{N/2 \text{ point DFT even index sequence}} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n) \cdot W_{\frac{N}{2}}^{nk}}_{N/2 \text{ point DFT odd index sequence}}$$

$$\therefore \boxed{X(K) = X_e(K) + W_N^k X_o(K)} \quad \text{--- (4)}$$

where $k = 0, 1, 2 \dots \frac{N}{2} - 1$.

When $K \geq \frac{N}{2}$ by Symmetry property $W_N^k = -W_N^{k+\frac{N}{2}}$

$$\therefore X(K) = X_e\left(K - \frac{N}{2}\right) - W_N^{k-\frac{N}{2}} \cdot X_o\left(k - \frac{N}{2}\right) \text{ for } k = \frac{N}{2}, \frac{N}{2}+1 \dots N-1 \quad \text{--- (5)}$$

for $\frac{N}{2}$ point DFT The complex Multiplications for $x_e(n)$ are $\left(\frac{N}{2}\right)^2$ and for $x_o(n)$ are $\left(\frac{N}{2}\right)^2$. i.e we require $2\left(\frac{N}{2}\right)^2$ Multiplications. These Two are Combined to $X(K)$. for this we require N Complex Multiplications.

$\therefore$ The total Number of Multiplications are

$$\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N = N + \frac{N^2}{2}. \quad \text{--- (6)}$$

$\therefore$ Total Number of Additions are

$$\frac{N}{2}\left(\frac{N}{2}-1\right) + \frac{N}{2}\left(\frac{N}{2}-1\right) + N = \frac{N^2}{2}. \quad \text{--- (7)}$$

The No. of Computations reduced by a factor 2. If we again Decompose the Sequence $X_e(k)$ & $X_o(K)$ into Two Subsequences. Then amount of Computation again be cut in half.

for Ex: let $N = 8$; Then $X_e(K)$ and $X_o(K)$ are 4-point DFT of Even & Odd index resp.

where $x_e(0) = x(0)$; $x_o(0) = x(1)$.
$x_e(1) = x(2)$; $x_o(1) = x(3)$
$x_e(2) = x(4)$; $x_o(2) = x(5)$
$x_e(3) = x(6)$; $x_o(3) = x(7)$.

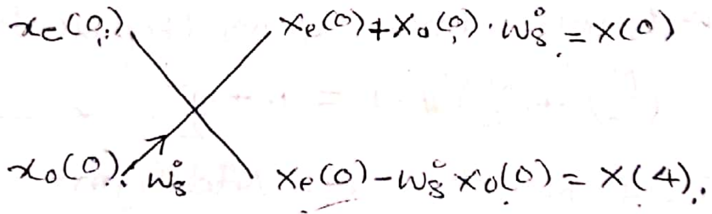
We have $X(K) = X_e(K) + W_8^K X_o(K)$ for $0 \le K \le 3$

$= X_e(K-4) - W_8^{K+4} X_o(\cancel{K-4})$ for $4 \le K \le 7$.

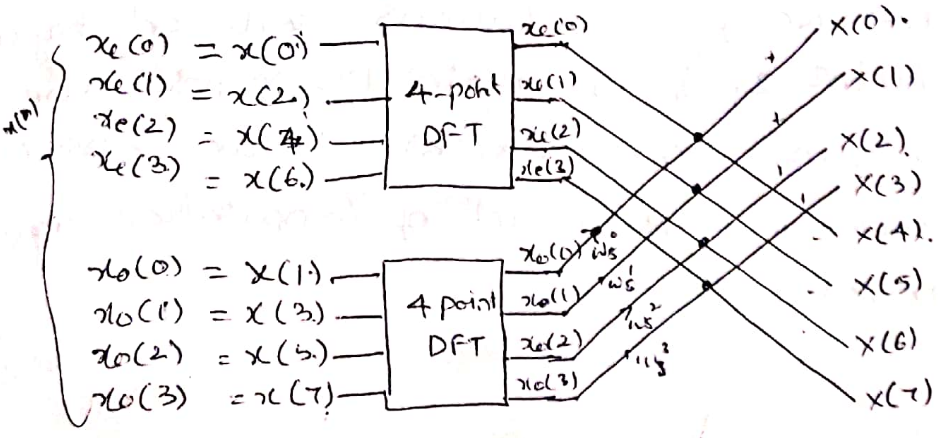
Then $X(0) = X_e(0) + W_8^0 X_o(0)$, $X(4) = X_e(0) - W_8^0 X_o(0)$;
$X(1) = X_e(1) + W_8^1 X_o(1)$; $X(5) = X_e(1) - W_8^1 X_o(1)$;
$X(2) = X_e(2) + W_8^2 X_o(2)$; $X(6) = X_e(2) - W_8^2 X_o(2)$;
$X(3) = X_e(3) + W_8^3 X_o(3)$; $X(7) = X_e(3) - W_8^3 X_o(3)$.

from Above we can observe that $X(0), X(4)$, $X(1), X(5)$, $X(2), X(6)$, $X(3), X(7)$ having same inputs.

∴ This operations can be represented by using Butterfly diagram as



The following diagram shows 8 point DFT by the use of two 4-point DFT



We can apply same Approach to Decompose each of $\frac{N}{2}$-point DFT. This canbe done By dividing sequence $x_e(n)$ & $x_o(n)$ into 2 seq consisting of Even and odd members of sequence.

so $\frac{N}{2}$-point DFT can be Expressed by $\frac{N}{4}$-point DFT.

$$\therefore X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k) \quad \text{for } 0 \le k \le \frac{N}{4}-1 \quad \text{— 9(a)}$$

$$= X_{ee}\left[k-\frac{N}{4}\right] + W_N^{2(k-\frac{N}{4})} \cdot X_{eo}\left(k-\frac{N}{4}\right) \text{ for } k \ge \frac{N}{4} \quad \text{— 9(b)}$$

similarly

$$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k) \quad \text{for } 0 \le k \le \frac{N}{4}-1 \quad \text{— 10(a)}$$

$$= X_{oe}\left(k-\frac{N}{4}\right) + W_N^{2(k-\frac{N}{4})} \cdot X_{oo}\left(k-\frac{N}{4}\right) \text{ for } k \ge \frac{N}{4} \quad \text{— 10(b)}$$

let for $N=8$, $\frac{N}{4}$ pt. gives 2 Even & 2 odd sequences

$x_{ee}(0) = x_e(0)$; $x_{oe}(0) = x_o(0)$

$x_{ee}(1) = x_e(2)$; $x_{oe}(1) = x_o(2)$

$x_{eo}(0) = x_e(1)$; $x_{oo}(0) = x_o(1)$

$x_{eo}(1) = x_e(3)$; $x_{oo}(1) = x_o(3)$

from Eq 9 we can write.

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$
$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$
$$X_e(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$$
$$X_e(3) = X_{ee}(1) - W_8^2 X_{eo}(1)$$

Same i/ps

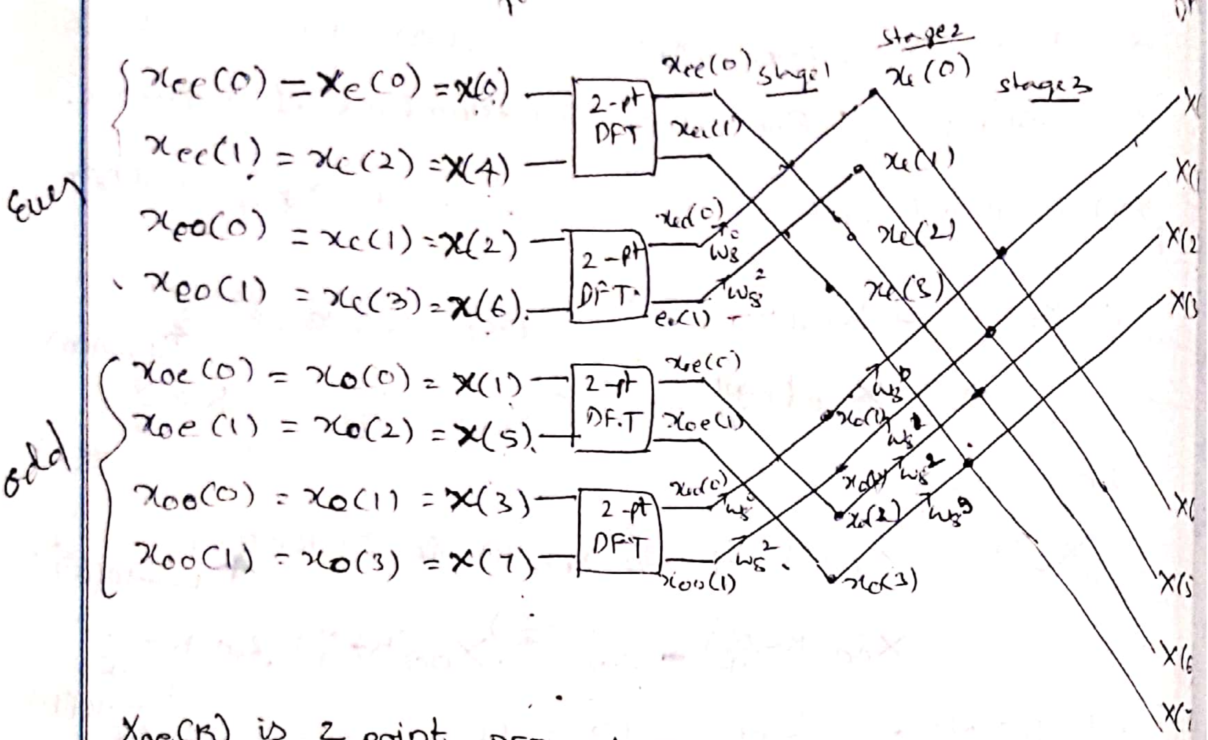
$0 \le k \le 1$

from eq 10 we can write

$$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$$
$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$
$$X_o(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$
$$X_o(3) = X_{oe}(1) - W_8^2 X_{oo}(1)$$

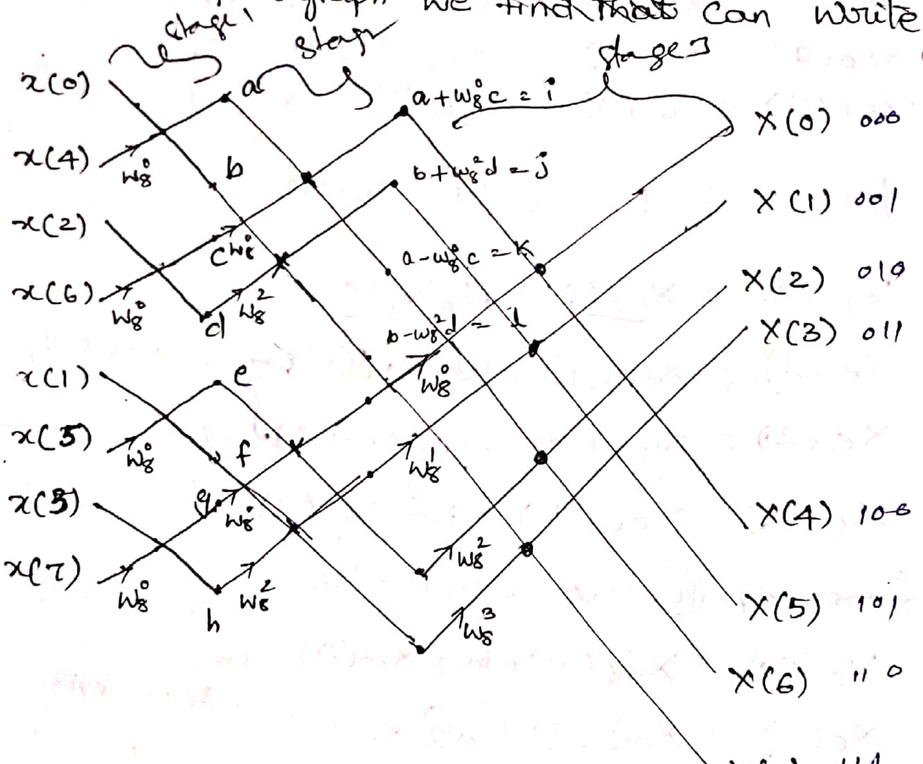
Same i/ps

These are Expressed by using Butterfly diagram as



$X_{oe}(K)$ is 2 point DFT of Even members of $x_o(n)$.
$X_{oo}(K)$ is 2 point DFT of Odd members of $x_o(n)$.

from Above flow graph we find that can write as



The Algorithm has been called Decimation in time Since at Each stage, The input sequence is divided into smaller sequences.

So, The total number of Complex Multiplications required to calculate DIT-FFT = $\frac{N}{2}\log_2 N$ And The total number of Complex Additions for Evaluating DFT using DIT-FFT is $N\log_2 N$.

$\therefore$ Speed improvement factor = $\dfrac{\text{No. of Complex Mult using Direct DFT}}{\text{No. of Complex Multi using FFT}}$

$$= \frac{N^2}{(N/2)\log_2 N}.$$

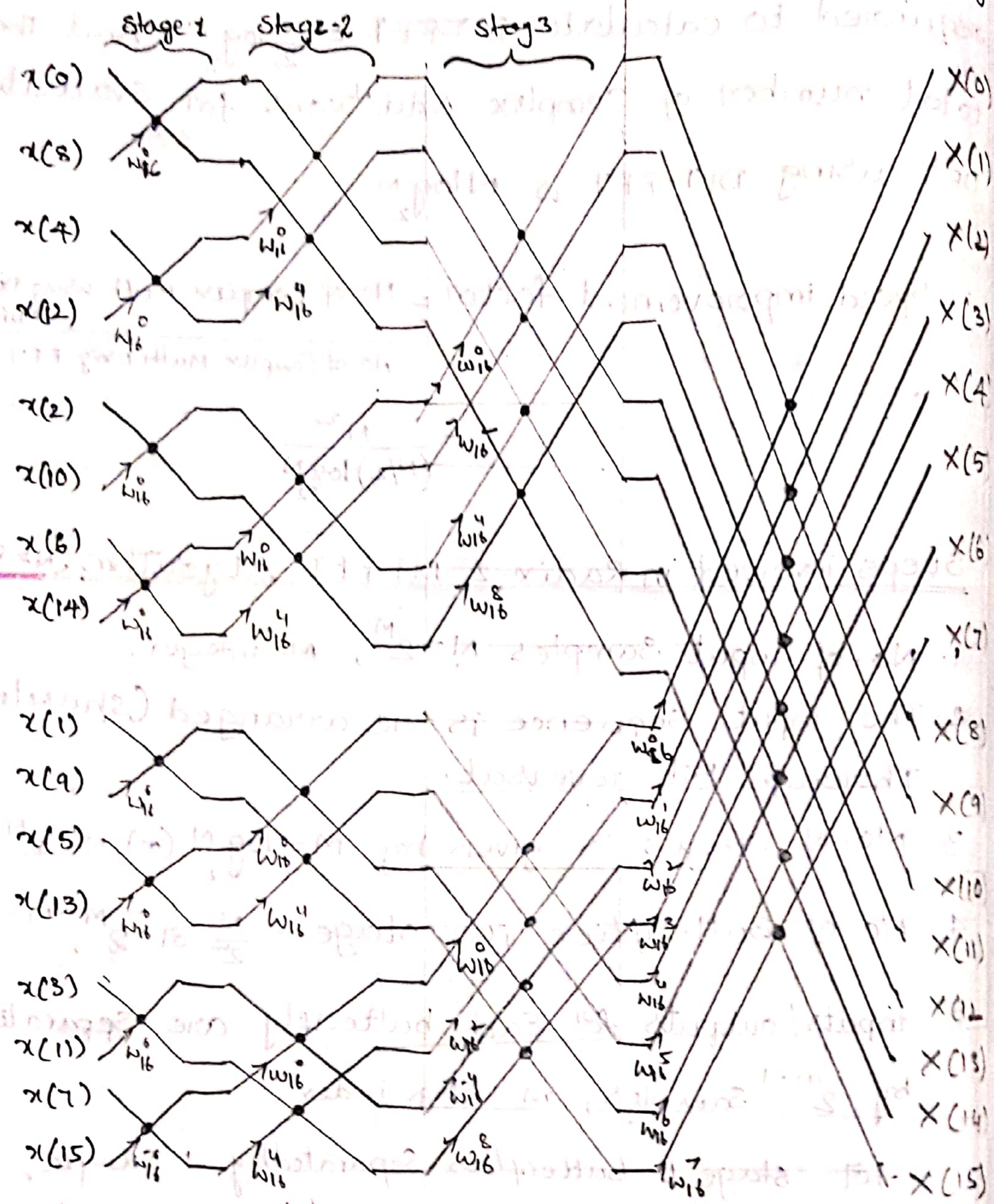
Steps involved in Radix-2 DIT-FFT Algorithm (N=8)

1. No. of input Samples $N = 2^M$, M - integer.
2. The input Sequence is re-arranged (shuffled) Through Bit-reversal.
3. No. of stages is given by $M = \log_2 N$ (or) $N = 2^M$
4. No. of Butterflies per stage = $\frac{N}{2^m}$ or 2^{M-m}. $\frac{N}{2^m} = 2^{M-m}$
5. inputs / outputs for Each butterfly are Separated by 2^{m-1} samples, m - stage index.

   for stage 1 butterflies Separated by 1 sample,
   stage 2 butterflies Separated by 2 Samples.
   stage 3 butterflies Separated by 4 Samples.
6. No. of Complex Multiplications given by $\frac{N}{2}\log_2 N$
7. No. of Complex Additions given by $N\log_2 N$.
8. The Exponent repeat factor (ERF) is given by 2^{M-m}.

* Exponent : A raised Symbol or Expression Beside a Numerical indicating how many times it is to be multiplied by itself.
Ex: $2^3 = 2\times2\times2$.

In This Similar Manner 16point DIT-FFT is given by



Ex: Compute 8-pt DFT of The sequence.

$$x(n) = \begin{cases} 1 & 0 \leq n \leq 7 \\ 0 & \text{otherwise} \end{cases}$$ by using DIT.

Sol: given sequence is $x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

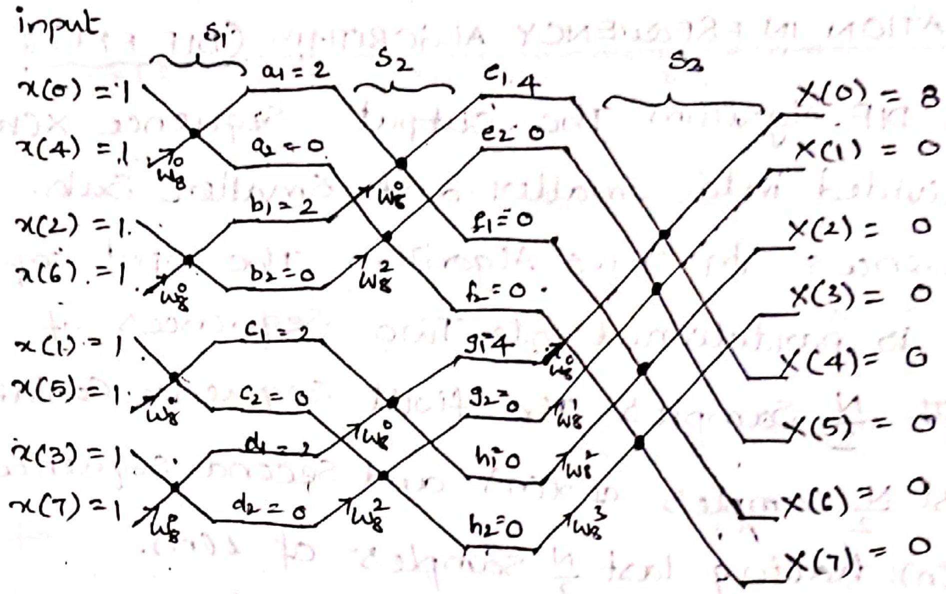
Twiddle factors associated with butterflies can be gives as

$W_8^0 = 1$

$W_8^1 = 0.707 - j0.707$

$W_8^2 = -j$

$W_8^3 = -0.707 - j0.707$



$\Rightarrow a_1 = 1 + W_8^0 1 = 1+1 = 2$; $a_2 = 1 - W_8^0 1 = 1 - 1 = 0$; $b_1 = 1 + W_8^0 1 = 2$; $b_2 = 1 - W_8^0 1 = 0$; $c_1 = d_1 = 2$; $c_2 = d_2 = 0$

$\Rightarrow e_1 = 2 + W_8^0 \cdot 2 = 2+2 = 4$; $f_1 = 2 - W_8^0 2 = 2-2 = 0$; $g_1 = 2 + W_8^0 2 = 2+2 = 4$; $h_1 = 2 - W_8^0 2 = 2-2 = 0$

$e_2 = 0 + W_8^2 0 = 0$; $f_2 = 0 - W_8^2 0 = 0$; $g_2 = 0 + W_8^2 0 = 0$; $h_2 = 0 - W_8^2 0 = 0$

$\Rightarrow X(0) = 4 + W_8^0 4$, $X(1) = 0 + W_8^1 \cdot 0$, $X(2) = 0 + W_8^2 \cdot 0$, $X(3) = 0 + W_8^3 0$

$X(4) = 4 - W_8^0 4$, $X(5) = 0 - W_8^1 0$, $X(6) = 0 - W_8^2 0$, $X(7) = 0 - W_8^3 0$

$\therefore X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

Ex 2: find DFT of $x_1(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIT-FFT Algorithm. also $x_2(n) = \{0, 1, -1, 0, 1, -1, 0, 0\}$

Ex 3: find 4-pt DFT of $x(n) = \{0, 1, 2, 3\}$ using DIT-FFT

Ex 4: find 8-pt DFT of $x(n) = \begin{cases} 1 & \text{for } 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$

Ex 5: find 8pt DFT of $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$ "

Ex 6: find 8pt DFT of $x(n) = 2^n$ where N=8 using DIT-FFT

Ex 7: find 4pt DFT of $x(n) = \cos\frac{n\pi}{2}$ using DIT-FFT

Ex 8: find 8pt DFT of (i) $x(n) = \{0.5, 0, 0.5, 0, 0.5, 0, 0.5, 0\}$

(ii) $x(n) = \{1, 2, 3, 1, 2, 3, 1, 2\}$ (iii) $\{1, -1, 1, -1, 1, -1, 1, -1\}$

(iv) $x(n) = 2$ for $0 \le n \le 3$ (v) $x(n) = n$ for $0 \le n \le 7$

DECIMATION IN FREQUENCY ALGORITHM (DIF-FFT):

In DIF Algorithm the output sequence $X(k)$ is divided into smaller and smaller sub-sequences. In this Algorithm the input seq. $x(n)$ is partitioned into two sequences of length $\frac{N}{2}$ samples. The first sequence contains first $\frac{N}{2}$ samples of $x(n)$ and second sequence $x_2(n)$ having last $\frac{N}{2}$ samples of $x(n)$.

i.e $x_1(n) = x(n)$, $n = 0, 1, 2 \ldots \frac{N}{2}-1$.

$x_2(n) = x(n+\frac{N}{2})$ $n = 0, 1, 2 \ldots \frac{N}{2}-1$.

If $N = 8$ then first sequence $x_1(n)$ is having values for $0 \le n \le 3$, & second $x_2(n)$ for $4 \le n \le 7$

The N-pt DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{nk}.$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x_2(n) \cdot W_N^{(n+\frac{N}{2})k}$$

$W_N^{\frac{Nk}{2}} = e^{-j\frac{2\pi}{N}\cdot\frac{Nk}{2}}$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{kn} + W_N^{\frac{Nk}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) \cdot W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + e^{-j\pi k} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{kn}$$

k – Even:

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{2nk} \quad \left[\because e^{-j\pi k} = 1 \text{ when k is even}\right]$$

$$\therefore X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n)+x_2(n)] W_N^{2nk}$$

$$= \sum_{n=0}^{N/2-1} [x_1(n)+x_2(n)] W_{N/2}^{nk} = \sum_{n=0}^{\frac{N}{2}-1} f(n) \cdot W_{N/2}^{nk} \quad \text{—— (1)}$$

where $f(n) = x_1(n) + x_2(n)$. —— (1b)

Let K—odd

$e^{-j\pi k} = -1$

$$X(2K+1) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)]\, W_N^{(2K+1)\cdot n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) - x_2(n)]\, W_N^{n} \cdot W_{N/2}^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g(n) \cdot W_{N/2}^{nk} \quad \text{——— (2a)}$$

where $g(n) = [x_1(n) - x_2(n)] \cdot W_N^{n}$ ——— 2(b)

for N = 8 from Eq (1a), (1,b).

$\left[W_8^4 = e^{-j\pi} = -1, \quad W_8^8 = e^{-j2\pi} = 1 \right]$

K=0 $X(0) = \sum_{n=0}^{3} f(n) = f(0) + f(1) + f(2) + f(3)$

K=1 $X(2) = \sum_{n=0}^{3} f(n) W_8^{2n} = f(0) + f(1) W_8^2 - f(2) - f(3) W_8^2$ $\left(\because W_8^6 = +W_8^{2+\frac{8}{2}} = -W_8^2 \right)$

K=2 $X(4) = \sum_{n=0}^{3} f(n) W_8^{4n} = f(0) - f(1) + f(2) - f(3)$

K=3 $X(6) = \sum_{n=0}^{3} f(n) W_8^{6n} = \sum_{n=0}^{3} f(n) (-W_8^2)^n$

$= f(0) - f(1) W_8^2 + f(2) + f(3) W_8^2$

$W_8^4 = e^{-j\frac{2\pi}{8}\cdot 4}$, $W_8^6 = W_8^{2+\frac{8}{2}} = -W_8^2$, $W_N^{n+\frac{N}{2}} = -W_N^n$

from Eq 2a, 2b.

K=0 $X(1) = \sum_{n=0}^{3} g(n) = g(0) + g(1) + g(2) + g(3)$

K=1 $X(3) = \sum_{n=0}^{3} g(n) \cdot W_8^{2n} = g(0) + g(1) W_8^2 - g(2) - g(3) \cdot W_8^2$

K=2 $X(5) = \sum_{n=0}^{3} g(n) \cdot W_8^{4n} = g(0) - g(1) + g(2) - g(3)$

K=3 $X(7) = \sum_{n=0}^{3} g(n)\, W_8^{6n} = \sum_{n=0}^{3} g(n) (-W_8^2)^n$

$= g(0) - g(1) \cdot W_8^2 - g(2) + g(3) W_8^2$

we have seen that Even-indexed samples of $X(k)$ can be obtained from 4-pt DFT of $f(n)$ where

$$f(n) = x_1(n) + x_2(n) \quad \text{where} \quad 0 \le n \le \frac{N}{2}-1$$

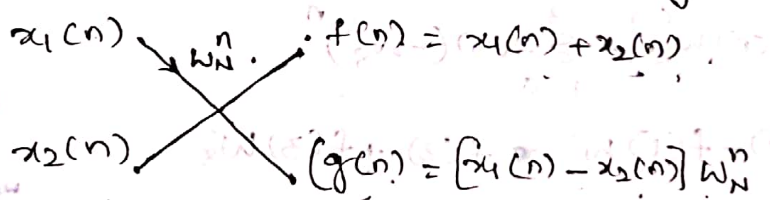
i.e
$$f(0) = x_1(0) + x_2(0)$$
$$f(1) = x_1(1) + x_2(1)$$
$$f(2) = x_1(2) + x_2(2)$$
$$f(3) = x_1(3) + x_2(3)$$

Odd-indexed samples of $X(k)$ obtained from $g(n)$

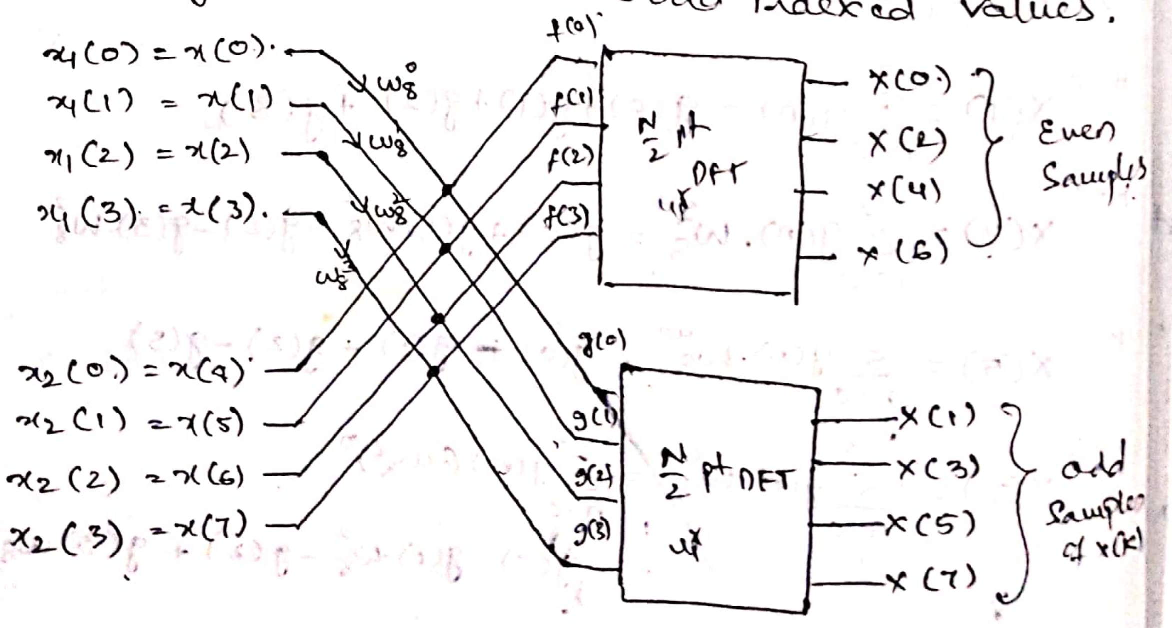
$$g(n) = [x_1(n) - x_2(n)]\, W_8^n$$

i.e
$$g(0) = [x_1(0) - x_2(0)]\, W_8^0$$
$$g(1) = [x_1(1) - x_2(1)]\, W_8^1$$
$$g(2) = [x_1(2) - x_2(2)]\, W_8^2$$
$$g(3) = [x_1(3) - x_2(3)]\, W_8^3.$$

for $f(n)$, $g(n)$ the butterfly diagram is given

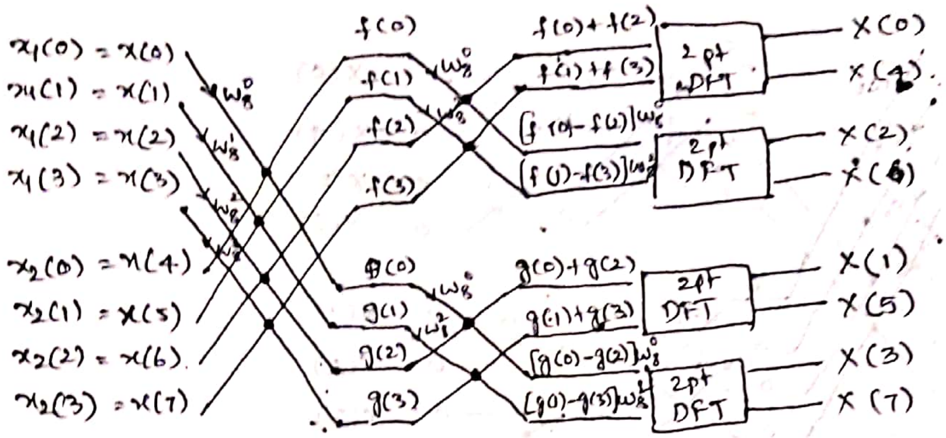


Similarly for above Even & odd indexed values.

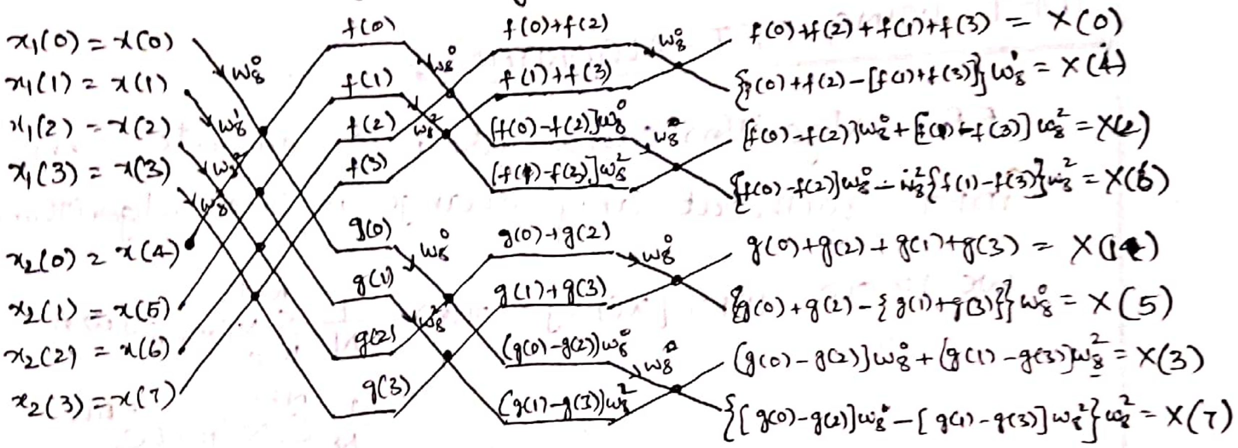


Scanned by CamScanner

After that we compute $\frac{N}{4}$-point DFTs. So above figure can be reduced as follows.



The above figure again reduced as follows.

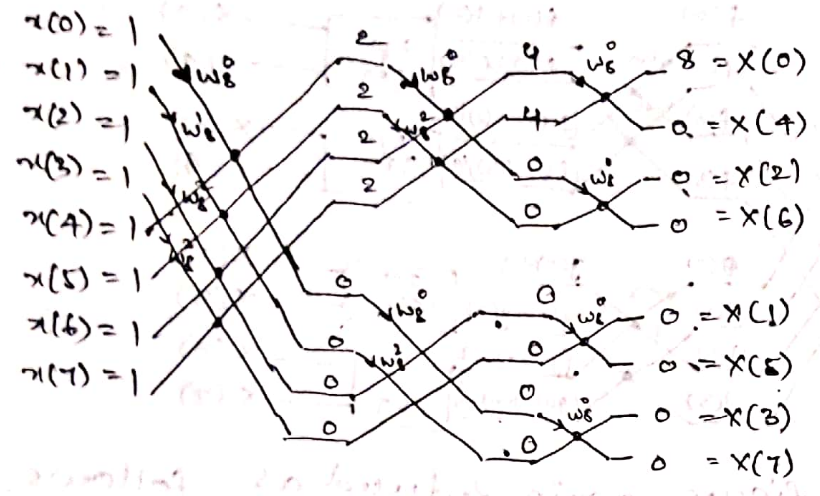


$\therefore$ This is the flow graph for 8-pt DFT.

STEPS REQUIRED TO FIND DFT USING RADIX-2 DIF-FFT.

1. No. of input samples $N = 2^M$, M - No. of stages
2. Input sequence is in Natural order
3. Each stage contains $\frac{N}{2}$ butterflies
4. Inputs/outputs for each butterfly are separated by 2^{M-m} samples, m - represents the stage-index.
5. The number of complex multiplications is given $\frac{N}{2}\log_2 N$.
6. The number of complex additions is $N\log_2 N$
7. The No. of Butterflies per stage is 2^{m-1}
8. ERF is given by 2^{m-1}.
9. Twiddle factor exponent is given by $K = \frac{Nt}{2^{M-m+1}}$

Ex: Compute 8-point DIF-FFT of sequence $x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$



IDFT USING FFT-ALGORITHM :-

FFT-Algorithm can be used to compute an IDFT without any change in the Algorithm.

We know $\text{IDFT}[X(k)] = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j\omega n}$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}$$

take Complex Conjugate & multiply by N

$$N \cdot x^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{nk} \quad \text{—— (1)}$$

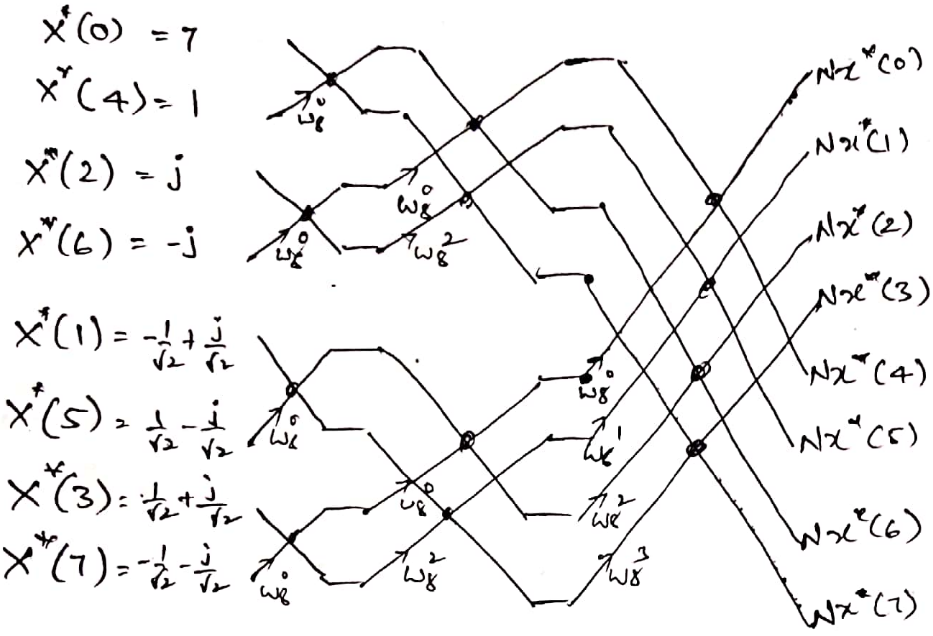
The Right side of Eq (1) is DFT of $X^*(k)$ and may be Computed using any FFT algorithm.

The desired o/p sequence can be obtained by Complex Conjugate the DFT & dividing by N

$$x(n) = \frac{1}{N} \left[\sum_{k=0}^{N-1} X^*(k) W_N^{nk} \right]^*$$

Ex: Compute IDFT of sequence using DIT Algorithm,

$X(K) = \{7, -0.707-j0.707, -j, 0.707-j0.707, 1, 0.707+j0.707, j, -0.707+j0.707\}$



$\therefore \quad N x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$

$x^*(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$

Ex: find IDFT of $X(K) = \{10, -2+j2, -2, -2-j2\}$ using DIT & DIF Algorithms.

Ex: find IDFT of following seqs. using (a) DIT (b) DIF

(i) $X(K) = \{1, 1+j, 1-j2, 1, 0, 1+j2, 1+j\}$

(ii) $X(K) = \{12, 0, 0, 0, 4, 0, 0, 0\}$

(iii) $X(K) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

(iV) $X(K) = \{8, 1+j2, 1-j, 0, 1, 0, 1+j, 1-j2\}$

(V) $X(K) = \{16, 1-j4.414, 0, 1+j0.414, 0, 1-j0.414, 0, 1+j4.41\}$