INTRODUCTION TO DSP

SIGNAL: It is defined as any physical quantity that varies with time, space or any other independent variable. Simply signal is called as Anything That carries some information.

ex: Speech, seismic, AC power supply, unemployment rate of Cntry., ECG EEG

* Signal can be function of one or more independent variables.

* If signal depends only on one variable, Then it is known as one dimensional signal.

ex: Speech, AC power supply

* If signal depends on Two independent variables it is known as Two-dimensional signal.

ex: X-ray images, sonograms.(image produced by ultra sound, a part taken from Human body)

Classification of signals:

a) Continuous-time signals: These are signals defined for every instant of time. Denoted by $x(t)$.

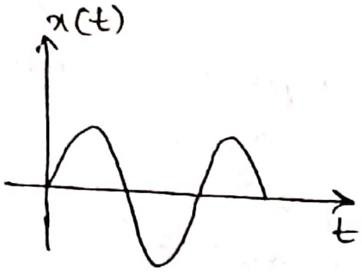
b) Descrete-time signals: These are signals defined at descrete instants of time. They are continuous in amplitude, but descrete in time. Denoted by $x(n)$.

c) Digital signals: These are descrete-in time and Quantized in amplitude.

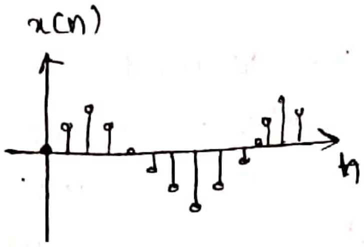
d) Deterministic signals: It is exhibiting no uncertainity of value at any given istant of time. Its instantaneous value can be accurately predicted by mathematical operation.

e) Random signal: It is characterized by uncertainity before its actual occurrence.

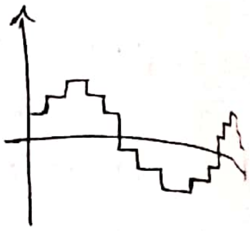
ex: Noise.



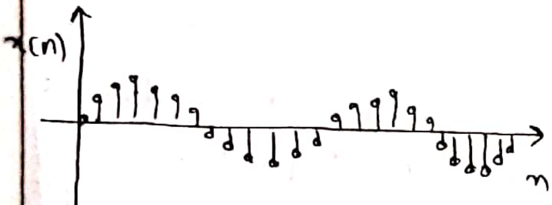
(a) Continuous-time Signal



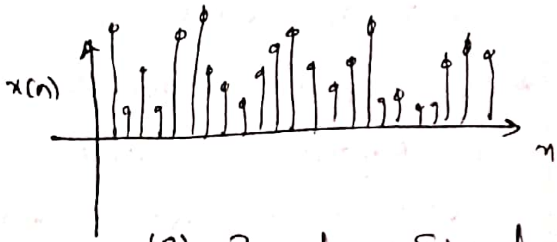
(b) Descrete-time signal



(c) Digital sig.



(d) deterministic Signal

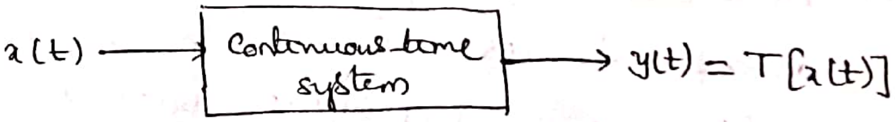


(e) Random Signal.

SYSTEM : It is defined as a physical device That generates a response or an output signal, for a giv input signal. It is interconnection of Components

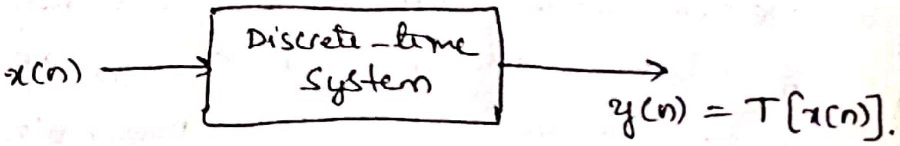
a) Continuous-time System:

It is a system which operates on a Continuou -time signal and produces a continuous-tim output Signal.



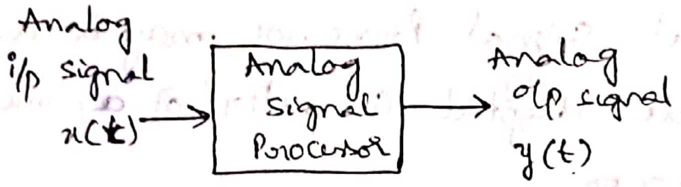
b) Discrete-time System :

It is a System which operates on a discrete -time ~~Sy~~ signal and produces a discrete-tim output Signal.



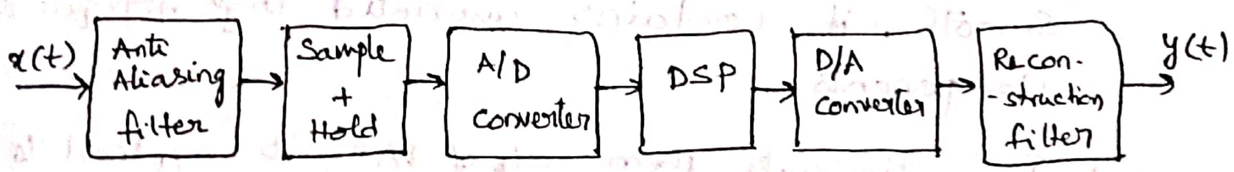
SIGNAL PROCESSING: It is any operation that changes the characteristics of a signal. These characteristics include the amplitude, shape, phase and frequency content of the signal.

Analog signal Processing system: The system that processes the analog signal is known as Analog signal Processing system.



Digital signal Processing system:

This system is divided into 6 sections.



→ The source of input signal is from a Transducer or a communication signal (ex. ECG, EEG).

→ Input signal is applied to Anti-aliasing filter which is lowpass filter used to remove the high frequency Noise and to band limit the signal.

→ In addition to the Lowpass filter, it may also include ~~The~~ a 50Hz notch filter That can remove the power frequency component, which is large part of external Noise.

→ The amplifier (Not shown in figure) may be used to bring signal upto the voltage range that is required for input of A/D converter.

→ The output of Sample and Hold circuit gives as input to the ADC. The output of ADC is an N-bit binary number depending on the value of Analog signal at its input.

→ The ADC signal is ~~tim~~ limited to a range of either 0 to +10V (if unipolar) or -5 to +5V if bipola

→ Once converted to digital form, the signal can be Processed using digital Techniques.

→ The digital signal Processor may be a large Programmable digital computer or a microprocess
ex: TMS 320C50

→ The o/p of DSP is applied to the input of a DAC.), This o/p of DAC is continuous but not Smooth, it contains unwanted High frequency components.

→ To eliminate them o/p of DAC is applied to a reconstruction filter which gives Smooth - continuous - time signal.

Advantages of DSP

(1) Greater Accuracy: The tolerance of circuit Comp used to design the Analog filters affects the Accu whereas dsp provides superior control of accu

(2) cheaper: In Many applications digital realiz is comparatively cheaper than its Analog count

(3) Ease of data storage: Digital signals can be easily stored on Magnetic Media without loss of fidelity and can be processed off-line in remote laboratory.

(4) Flexibility in Configuration: A DSP system can be easily reconfigured by changing the program. But in Analog system reconfiguration involves the redesign of system to hardware.

(5) Applicability of VLF signals: ~~Using~~ Using DSP very low frequency signals such as those occuring in seismic application can be easily processed.

(6) Time sharing: DSP allows sharing of a given Processor among a number of signals by time sharing, Thus reduces the cost of processing a signal.

Limitations:

(1) System Complexity: System complexity increases in the digital Processing of an Analog signal because of devices such as A/D and D/A Converters and Their associated filters.

(2) Bandwidth limited by Sampling rate: Band limited signal can be sampled without information loss if The sampling rate is more than twice the Bandwidth. Therefore signals having extremely wide Bandwidths, require fast sampling rate A/D Converters and fast digital signal Processors. But There is Practical limitation in the Speed of operation of A/D Converters and DSPs.

(3) Power Consumption: A variety of Analog Processing algorithms can be implemented using passive ckt elements like inductors, capacitors and resistors That do not need much power, where as a DSP chip containing over a 4 lakh Transistors - dissipates more power (1 Watt)

Applications of DSP :

1. Telecommunication: Echo cancellation in Telephone N Telephone dialling application, Modems, Line repe channel multiplexing, Data encryption ; ~~ve~~ video -c -rencing, Cellular phone, FAX.

2. Consumer Electronics:
Digital Audio/TV, electronic music synthesizer, educational toys, FM stereo applications, Sound recording appli-cations.

3 Instrumentation and control:
Spectrum Analysis, Digital filter, PLL, Function - Generator, Servo control, Spectrum Analysis, Robot Control, Process control.

4. Image processing :
Image compression, Image Enhancement, Image Analysis and recognition.

5. Medicine :
Medical diagnostic instrumentation such as Comput -rized Tomography (CT), x-ray scanning, Magnetic resonance imaging, Spectrum Analysis of ECG & EE signals to detect the various disords in Heart ai Brain, patient monitoring.

6. Speech Processing: used in Automatic speech recogni Speaker verification and identification.

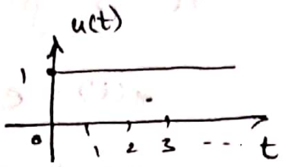
7. Seismology: DSP Techniques are ~~so~~ employed in T geophysical exploration for oil and gas, detection o underground Nuclear explosion and earthquake monitoring.

8. Military: Radar signal processing, Sonar Sign Processing, Navigation, Secure Communications,

Elementary Continuous-time Signals:

ⓐ Unit-step function:

It is defined as $u(t) = 1$ for $t \geq 0$
$= 0$ for $t < 0$



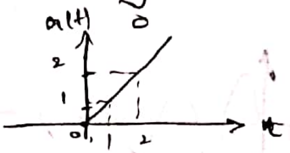
ⓑ Unit-Ramp function:

It is defined as $r(t) = t$ for $t \geq 0$
$= 0$ for $t < 0$

$r(t) = t \cdot u(t)$

we can obtained as $r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} d\tau = t, (t \geq 0)$

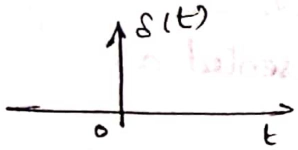
$\Rightarrow u(t) = \frac{dr(t)}{dt}.$



ⓒ Impulse Function:

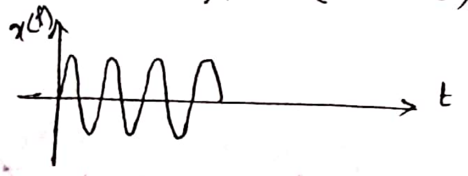
The unit impulse function defined as, $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

and $\delta(t) = 0$ for $t \neq 0$.



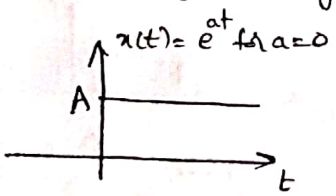
(d) Sinusoidal Signal:

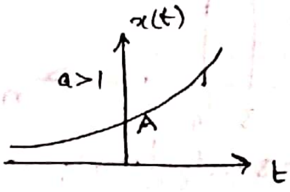
$x(t) = A \sin(\Omega t + \theta)$ where A - amplitude, Ω - freq, θ - phase
in radians

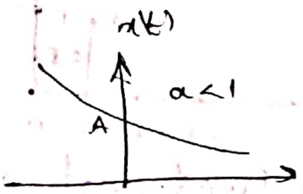


(e) Real Exponential Signal:

it is given by $x(t) = A \cdot e^{at}$







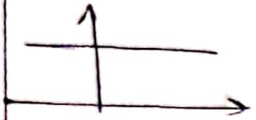
(f) Complex Exponential signal:

It is given by $x(t) = e^{st}$ where $s = \sigma + j\Omega$.

$$x(t) = e^{st} = e^{(\sigma+j\Omega)t} = e^{\sigma t} \cdot e^{j\Omega t}$$

we know $e^{j\Omega t} = \cos\Omega t + j\sin\Omega t$

$$x(t) = e^{\sigma t}\left[\cos\Omega t + j\sin\Omega t\right]$$

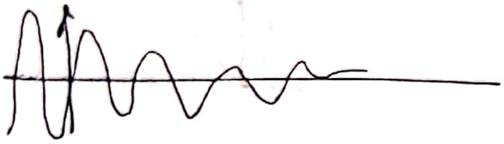


$s=0$

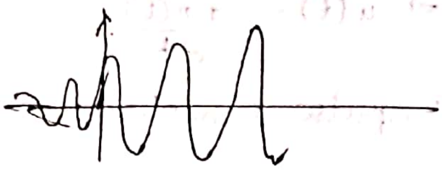
$s=-\sigma$

$s=\sigma$

$s=\pm j\Omega$



$s=\sigma\pm j\Omega ; \sigma<0$



$s=\sigma\pm j\Omega , \sigma>0$

Representation of Discrete-time Signals:

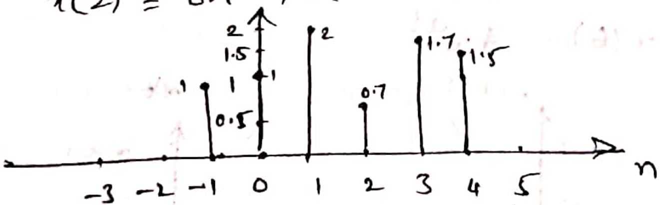
Discrete-time Signals can be represented as

- ⓐ Graphical representation.
- ⓑ Functional representation.
- ⓒ Tabular representation,
- ⓓ Sequence representation.

ⓐ Graphical representation:

Let $x(n)$ is signal with values $x(-1) = 1, x(0) = 1, x(1) =$

$x(2) = 0.7, x(3) = 1.7.$



ⓑ Functional representation:

$$x(n) = \begin{cases} 1 & \text{for } n=-1,0 \\ 2 & \text{for } n=1 \\ 0.7 & \text{for } n=2 \\ 1.7 & \text{for } n=3 \\ 1.5 & \text{for } n=4 \end{cases}$$

ⓒ Tabular representation:

The discrete time signal can be represented as

n	−1	0	1	2	3	4
x(n)	1	1	2	0.7	1.7	1.5

ⓓ Sequence representation:

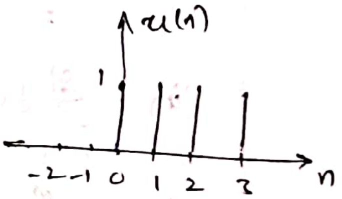
$x(n) = \{1, \underset{\uparrow}{1}, 2, 0.7; 1.7, 1.5\}$ finite sequence

$x(n) = \{ \ldots .\ 0.2, \underset{\uparrow}{1}, -1, 3, 2, 5 \ldots \}$ infinite sequence

Elementary Discrete-time Signals:

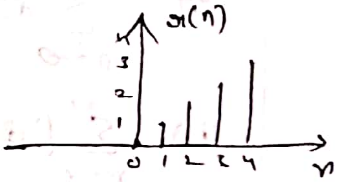
ⓐ Unit-step sequence:

It is defined as $u(n) = 1$ for $n \geq 0$
$= 0$ for $n < 0$



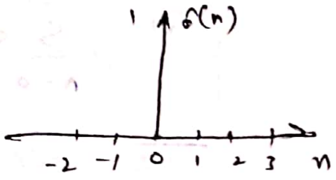
ⓑ Unit ramp sequence:

It is defined as $u_r(n) = n$ for $n \geq 0$
$= 0$ for $n < 0$



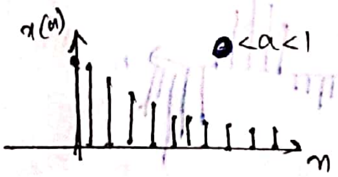
ⓒ Unit sample or impulse sequence:

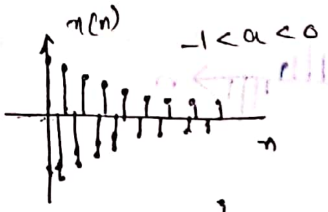
It is defined as $\delta(n) = 1$ for $n = 0$
$= 0$ for $n \neq 0$

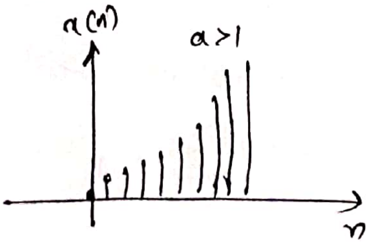


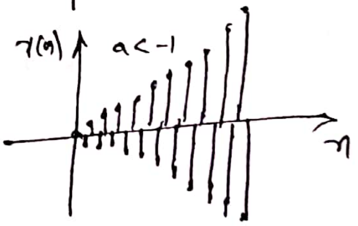
ⓓ Exponential sequence:

It is defined as $x(n) = a^n$ for all n



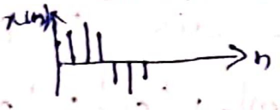






ⓔ Sinusoidal signal:

Discrete-time sinusoidal Signal is $x(n) = A\cos(\omega_0 n + \phi)$



ⓕ Complex Exponential Signal:

It is defined as $x(n) = a^n . e^{j(\omega_0 n + \phi)}$

$$= a^n \left[\cos(\omega_0 n + \phi) + j \sin(\omega_0 n + \phi) \right]$$

for $|a|=1$, the sequence is sinusoidal, for $|a|<1$, the amplitude of sinusiodal sequence decays exponential and for $|a|>1$ the amplitude of sinusoidal sequence increases exponentially.

Examples:

find following Summations.

(i) $\sum_{n=-\infty}^{\infty} \delta(n-2)\sin 2n$ $\left[\because \delta(n-2) = 1 \text{ for } n=2 \atop = 0 \text{ for } n \neq 2\right]$

$= \sin 2n \Big|_{n=2} = \sin 4$

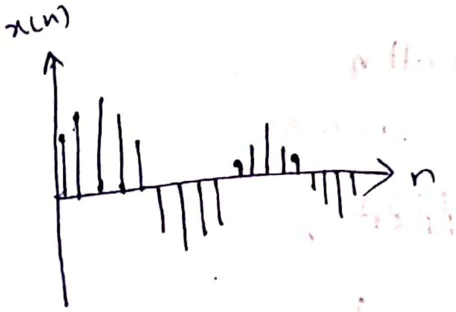
(ii) $\sum_{n=0}^{\infty} \delta(n) e^{2n} = e^{2n}\Big|_{n=0} = 1$

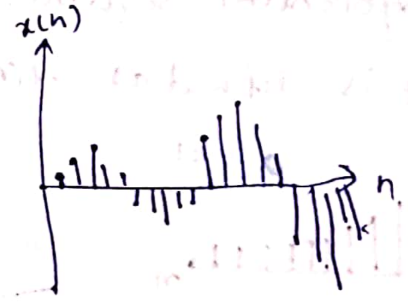
(iii) $\sum_{n=-\infty}^{\infty} \delta(n+1) x(n) = x(n)\Big|_{n=-1} = x(-1)$

(iv) $\sum_{n=-\infty}^{\infty} \left[\delta(n-2)\cos 2n + \delta(n-1)\sin 2n\right]$

$= \cos 4 + \sin 2$

(v) $\sum_{n=0}^{\infty} \delta(n+1) e^{-2n} = 0$.





Classification of Discrete time signals:

Discrete time signals are classifiede as

ⓐ Energy signals and Power signals.

ⓑ Periodic and Aperiodic signals.

ⓒ Symmetric and Anti Symmetric signals

ⓓ Causal and Non-Causal signals.

ⓐ Energy signals and $\overset{\text{power}}{\text{~~Periodic~~}}$ signal.

For a discrete-time signal $x(n)$

The Energy signal is defined as $\boxed{E = \sum_{n=-\infty}^{\infty} |x(n)|^2}$

The average power of $x(n)$ is $\boxed{P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$

* A signal is Energy signal if and only if Total Energy of signal, $\boxed{E - \text{finite}, \text{Power } P = 0}$

* A signal is Power signal if and only if Average power of signal $\boxed{P - \text{finite}, \text{Energy } E = \infty}$

* If signal not satisfy above properties are ~~neigh~~ neither energy nor power signals.

Example 1:

Find the following signals are either Energy or power signals.

(i) $x(n) = (1/3)^n u(n)$

Total Energy of $x(n)$ is $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |(1/3)^n u(n)|^2$

$$E = \sum_{n=0}^{\infty} [(1/3)^n]^2 \quad \left[\because u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}\right]$$

$$\left(\because \sum_{n=0} a^n = \frac{1}{1-a}\right)$$

$$E = \sum_{n=0}^{\infty} \left[\frac{1}{9}\right]^n \qquad \left[\because 1+a+a^2+\dots\infty = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-1/9} = 9/8$$

$$\therefore \boxed{E = 9/8 = \text{finite}}$$

Average power of $x(n)$ is $P = \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=0}^{N} \left[1/9\right]^n$

$$\therefore P = \lim_{N\to\infty} \frac{1}{2N+1}\left[\frac{1-(1/9)^{N+1}}{1-1/9}\right] \qquad \left[\because \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \text{ for } a\neq 1\right]$$

$\boxed{P = 0}$ Hence signal is Energy signal.

Example: 2

$$x(n) = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}$$

Energy, $E = \sum_{n=-\infty}^{\infty} \left|e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\right|^2 \qquad \left[\because e^{j(\omega t+\theta)} = 1\right]$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty \qquad \boxed{\therefore E = \infty}$$

Power, $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|e^{j\left(\frac{\pi}{2}n+\frac{\pi}{4}\right)}\right|^2$

$$= \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \underbrace{\sum_{n=-N}^{N} 1}_{\cancel{2N+1}} \qquad \left[\because \sum_{n=-N}^{N} 1 = 2N+1\right]$$

$\boxed{P = 1}$ = finite

Hence $x(n)$ is power signal.

Example: 3

$$x(n) = \sin\left(\frac{\pi}{4}n\right)$$

Energy, $E = \sum_{n=-\infty}^{\infty} \left|\sin^2\left(\frac{\pi}{4}\right)n\right|$

$$= \sum_{n=-\infty}^{\infty} \frac{1-\cos 2\left(\frac{\pi}{4}n\right)}{2} = \infty$$

$$\boxed{\therefore E = \infty}$$

power, $P = \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} \sin^2(\pi/4)\, n$

$$= \frac{1}{2} \cdot \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} 1 = 1/2$$

$\therefore \boxed{P = 1/2}$ = finite

Hence signal is Power signal.

Example 4:

$x(n) = e^{2n} u(n)$

$$\text{Energy, } E = \sum_{-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} e^{4n} = 1 + e^4 + e^8 + \cdots \infty = \infty$$

$$\boxed{\therefore E = \infty}$$

$$\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot \sum_{n=0}^{N} e^{4n}$$

$$= \operatorname{lm}_{N\to\infty} \frac{1}{2N+1} \cdot \left[\frac{e^{4(N+1)} - 1}{e^4 - 1} \right] = \infty$$

$$\boxed{\therefore P = \infty}$$

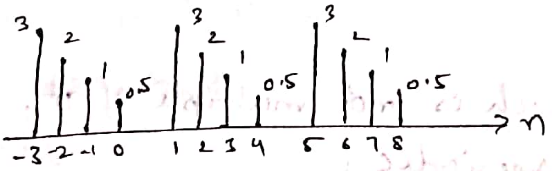
Hence

The signal is neither power nor energy signal

Periodic and Aperiodic Signals:

A discrete time ~~expe~~ signal, ~~FN~~ $x(n)$ is said to be periodic with period 'N' if and only if $\boxed{x(n+N) = x(n)}$ for all 'n'.

ex:



otherwise The signal is called as Aperiodic

ex:

let $x(n) = A \cdot \sin(\omega_0 n + \phi)$

$x(n+N) = x(n)$

$x(n+N) = A \sin(\omega_0 (n+N) + \theta)$

$= A \sin(\omega_0 n + \omega_0 N + \theta)$

it satisfy above rules, Then it is periodic

$x(n+N) = A\sin(w_0(n+N)+\theta)$

$= A\sin(w_0 n + w_0 N + \theta)$

it satisfies if and only if $w_0 N$ is an integer multiple of 2π. i.e

$$w_0 N = 2\pi m$$

$$\Rightarrow \boxed{w_0 = \frac{2\pi m}{N}}$$

$$\Rightarrow \quad N = 2\pi\left(\frac{m}{w_0}\right)$$

where N is called fundamental Period.

Example-I : Find the following signals are either periodic or aperiodic? and calculate fundamental Periods.

$x(n) = e^{j6\pi}$.

The fundamental frequency $w_0 = 6\pi$. it is multiple of π so signal is Periodic.

$$N = 2\pi\left[\frac{m}{w_0}\right] = 2\pi\left[\frac{m}{6\pi}\right] = 1 \quad (\because m=3)$$

Example 2:

$x(n) = e^{j3/5[n+1/2]}$

Here $w_0 = 3/5$, which is not multiple of π.

so signal is Aperiodic

Example 3:

$x(n) = \cos\frac{2\pi}{3}n$

Here $w_0 = \frac{2\pi}{3}$, which is multiple of π

so signal is Periodic

Fundamental period $N = \frac{2\pi m}{w_0} = \frac{2\pi m}{2\pi/3} = 3m = 3 \ (m=1$

Example-4:

$x(n) = \cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$ signal is periodic

The fundamental period of $\cos(\frac{\pi}{3}n)$ is

$$N_1 = \frac{2\pi m}{\pi/3} = 6 \text{ for } m=1$$

Similarly for $\cos\frac{3\pi}{4}n$ is $N_2 = \frac{2\pi m}{3\pi/4} = 8$ for $m=3$

$$\therefore \frac{N_1}{N_2} = \frac{6}{8} = 3/4 \Rightarrow 4N_1 = 3N_2$$

Hence overall period $N = 4N_1 = 3N_2 = 12$

Example:5:

$x(n) = \sin\frac{\pi n}{4}$, $\omega_0 = \pi/4$ multiple of π

signal is periodic

$$N = \frac{2\pi m}{\pi/4} = 8 \text{ for } m=1$$

Example-6:

$x(n) = e^{j2n}$

$\omega_0 = 2$, which is not a multiple of π

So signal is aperiodic

~~$N = 2\pi m$~~

Example-7:

$x(n) = \cos\frac{\pi}{4}n + \cos 2n$

signal is Aperiodic

ⓒ Symmetric (Even) and Anti-Symmetric (odd) signals

→ A ~~Sa~~ discrete time signal $x(n)$ is said to be a Symmetric (Even) signal if it satisfies the condition $x(-n) = x(n)$ for all n. —①

ex: $x(n) = \cos wn$

→ The signal is ~~to b~~ said to be odd signal if it satisfies the condition

$$x(-n) = -x(n) \text{ for all } n. \quad \text{—②}$$

ex: $x(n) = A \cdot \sin wn$

→ A Signal $x(n)$ can be expressed as sum of even and odd Components.

i.e $x(n) = x_e(n) + x_o(n)$ ———③

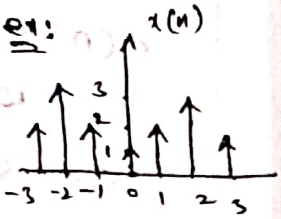
Then from Eq ②&③ $x(-n) = x_e(-n) + x_o(-n)$

$$x(-n) = x_e(n) - x_o(n) \quad \text{———④}$$

③+④ ⇒ $x_e(n) = \dfrac{x(n) + x(-n)}{2}$

③-④ ⇒ $x_{od}(n) = \dfrac{x(n) - x(-n)}{2}$

ex:



ⓓ Causal and Non-Causal Signals:

A signal is said to be causal if its value is zer for $n < 0$, otherwise signal is Non-causal.

ex: $x_1(n) = a^n u(n)$
$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$ } Causal Signals

ex: $x_1(n) = a^n u(-n+1)$
$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$ } Non-Causal Signals.

Operation on Signals:

The basic set of operations that are applying on signals to process are

a) Shifting (b) Time reversal (c) Time scaling

d) Scalar Multiplication (e) Signal Multiplier (f) Signal Addition

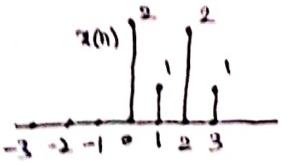
a) **Shifting**: The shift operation takes the input sequence and shifts the values by an integer increment of the independent variable. The shifting may delay or advance the sequences in time.

ex: if $x(n)$ is input & $y(n)$ o/p. Then $y(n) = x(n-k)$.

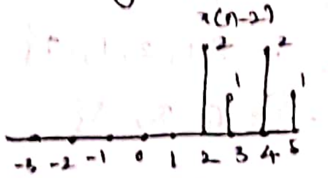
* k is +ve, shifting delays the sequence

* k is -ve, shifting Advances the sequence.

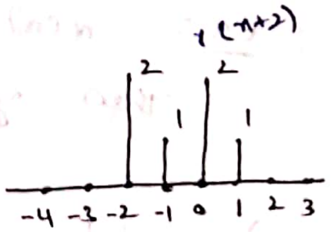
ex:
$x(n)$ is sequence Then $x(n-2)$ delayed by 2 units, $x(n+2)$ is Advance by 2 units.



ⓐ input



ⓑ delayed by 2

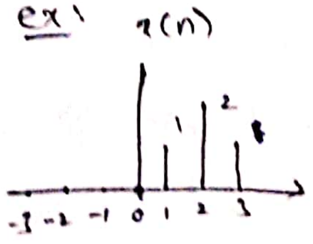


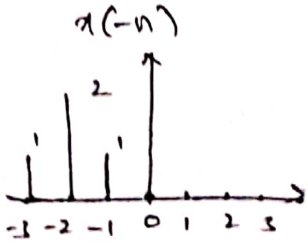
ⓒ Advanced by 2

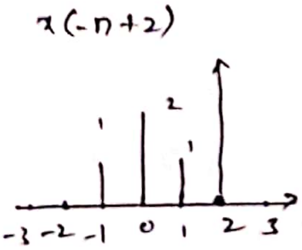
ⓑ Time Reversal:

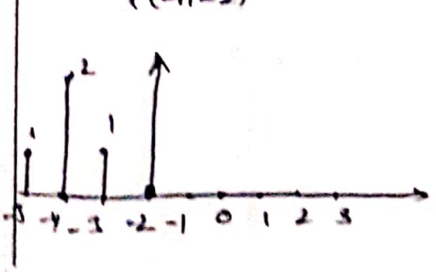
Time Reversal of sequence $x(n)$ can be obtained by folding sequence about $n=0$. It is Denoted as $x(-n)$

ex:





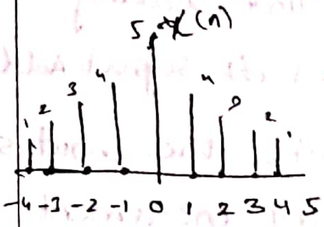


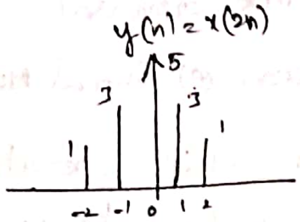


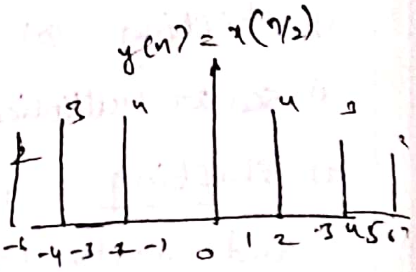
3) Time Scaling :

This is accomplished by replacing 'n' by 'λn' in the Sequ
$x(n)$.

ex: If $\lambda = 2$, $y(n) = x(2n)$

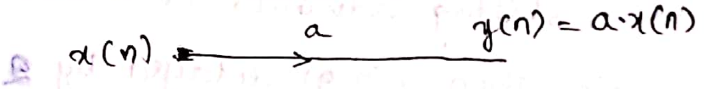






If we ~~time~~ scaled the time by a factor Then if multipl with scalar, The interval reduces. if it is divided th interval increases.

4) Scalar multiplication:

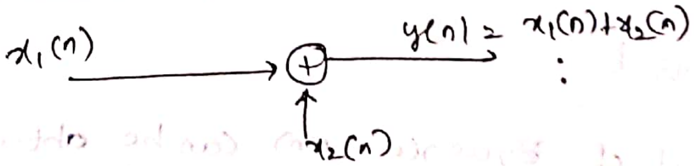


ex: $x(n) = \{1, 2, 1, -1\}$ let $a = 3$

Then $y(n) = 3 \times \{1, 2, 1, -1\}$

$= \{3, 6, 3, -3\}$

5) Addition operation:



$x_1(n) = \{1, 2, 3, 4\}$, $x_2(n) = \{4, 3, 2, 1\}$

Then $x_1(n) + x_2(n) = \{5, 5, 5, 5\}$

Classification of Discrete-time Systems.

Discrete-time systems are classifide as following types according to Their general properties and charactoristics as.

1. static and Dynamic Systems.
2. Causal and Non-Causal Systems.
3. Linear and Non-linear systems.
4. Time variant and Time-invariant systems.
5. FIR and IIR Systems.
6. Stable and Unstable Systems.

(1) Static and Dynamic Systems:

→ A Discrete-time System is called static or Memoryless if its output at any instant 'n' depends on the input samples at the same time, but not on past or future samples of the input. In other case system is called Dynamic.

ex:

$$\left.\begin{aligned} y(n) &= ax(n) \\ y(n) &= ax^2(n) \\ y(n) &= x(n)+x^2(n) \end{aligned}\right\} \text{Static} \qquad \left.\begin{aligned} y(n) &= x(n-1)+x(n-2) \\ y(n) &= x(n+1)+x(n) \\ y(n) &= x(n)\cdot x(n-1) \end{aligned}\right\} \text{Dynamic}$$

(2) Causal and Non-causal Systems:

→ A system is said to be causal if The output of The system at any time 'n' depends only at present and past inputs, but does not depend on future i/ps.

i.e $y(n) = F[x(n), x(n-1), x(n-2) \cdots]$

→ If o/p depends on future inputs, The system is said to be Non-causal or anticipatory.

ex:

$$\left.\begin{aligned} y(n) &= x(n)+x(n-1) \\ y(n) &= x(n)+\frac{1}{x(n-1)} \\ y(n) &= Ax(n)+B \\ y(n) &= ax(n)+bx(n-1) \end{aligned}\right\} \text{Causal system} \qquad \left.\begin{aligned} y(n) &= x(n^2) \\ y(n) &= x(2n) \end{aligned}\right\} \text{non Causal System}$$

(3) Linear and Non-linear Systems:

→ If a System Satisfies Superposition Principle, Then it is said to be Linear System, otherwise called as Non-linear System.

→ The Superposition Principle states that, The response of the System to a weighted Sum of signals should be Equal to The corresponding weighted sum of o/ps of the System to each of The individual input Signals.

→ A System is Linear if and only if

$$\boxed{T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]}$$

where a_1, a_2 - arbitrary Constants

ex: Determine which of The following are linear

ⓐ $y(n) = x(n) + \frac{1}{x(n-1)}$

for Two i/p sequences The o/ps are

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)} \quad \text{——} ①$$

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)} \quad \text{——} ②$$

Let $y_3(n)$ is o/p due to weighted sum of Two i/ps

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)]$$

$$\Rightarrow y_3(n) = a_1x_1(n) + a_2x_2(n) + \frac{1}{a_1x_1(n-1) + a_2x_2(n-1)} \quad \text{——} ③$$

from ① & ②

$$a_1y(n) + a_2y_2(n) = a_1x_1(n) + \frac{a_1}{x_1(n-1)} + a_2x_2(n) + \frac{a_2}{x_2(n-1)} \quad \text{—}$$

Hence Eq ③ ≠ Eq ④, doesn't satisfies Superposite
Princip

∴ system is Non-linear.

(ii) $y(n) = x^2(n)$

o/ps due to Two input signals $x_1(n), x_2(n)$ are

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T(x_2(n)] = x_2^2(n)$

weighted Sum of o/ps given by

$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1^2(n) + a_2 x_2^2(n)$ —①

The o/p due to weighted Sum of i/ps is

$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = [a_1x_1(n) + a_2x_2(n)]^2$ —②

from ①, ② $\{① \neq \{②$

∴ It doesnt Satisfies Superposition Principle.

Hence System y(n) is Non-Linear.

(iii) $y(n) = n.x(n)$

o/ps due to two i/p signals $x_1(n)$ & $x_2(n)$ are

$y_1(n) = T[x_1(n)] = n \cdot x_1(n)$ —①

$y_2(n) = T[x_2(n)] = n \cdot x_2(n)$ —②

The weighted Sum of o/ps given by

$a_1 y_1(n) + a_2 y_2(n) = a_1 T[x_1(n)] + a_2 T[x_2(n)]$

$= a_1 \cdot n \cdot x_1(n) + a_2 \cdot n \cdot x_2(n)$ —③

The o/p Due to weighted Sum of i/ps is

$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = a_1 \cdot n \cdot x_1(n) + a_2 \cdot n \cdot x_2(n)$ —④

Eq ③ = Eq ④ , satisfies superposition Principle

Hence system is Linear.

④ Time-variant and in-variant systems:

→ A system is said to be time-invariant or shift-invariant if the characteristics of the system donot change with time.

→ for a time-invariant system if $y(n)$ is the response of the system to i/p $x(n)$, then response of the system to the input $x(n-k)$ is $y(n-k)$.

If i/p is delayed by K samples then o/p is

$y(n,K) = T[x(n-K)]$

Delay the o/p sequence by K-samples, denoted as $y(n-K)$ then $\boxed{y(n,K) = y(n-K)}$ for all K values

Then system is time-invariant

→ If $\boxed{y(n,K) \neq y(n-K)}$ for all 'K' system is time-variant.

Note: A LTI system satisfies both the linearity and the time-invariant properties.

Examples:

(i) $y(n) = x(n) + x(n-1)$

If i/p is delayed by K units in time, then

$y(n,K) = T[x(n-K)] = x(n-K) + x(n-K-1)$

If o/p is delayed by K units in time, then

$y(n-K) = x(n-K) + x(n-K-1)$

Here $y(n-K) = y(n,K)$

so system is time-invariant

(ii) $y(n) = x(-n)$

i/p delayed by k units in time, then

$$y(n,k) = T[x(n-k)] = x(-n-k).$$

o/p delayed by k units in time

$$y(n-k) = x(-(n-k)] = x(-n+k)$$

$$\therefore \boxed{y(n,k) \neq y(n-k)}$$

Hence system is Time-variant.

(iii) $y(n) = x\left(\frac{n}{2}\right)$

i/p is delayed by k units in time, then

$$y(n,k) = T[x(n-k)] = x\left(\frac{n}{2}-k\right)$$

o/p is delayed by k units in time

$$y(n-k) = x\left(\frac{n-k}{2}\right)$$

$\therefore$ $y(n,k) \neq y(n-k)$ system is Time-variant.

(iv) $y(n) = n \cdot x^2(n)$

i/p is delayed by k units in time, then

$$y(n,k) = T[x(n-k)] = n(x^2(n-k))$$

o/p is delayed by k units in time.

$$y(n-k) = (n-k) \cdot x^2(n-k)$$

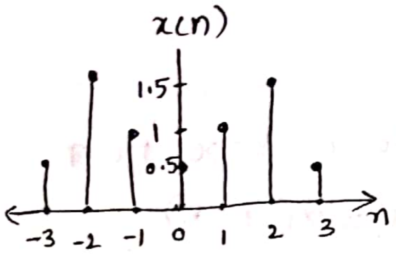
$y(n,k) \neq y(n-k)$. system is Time-variant.

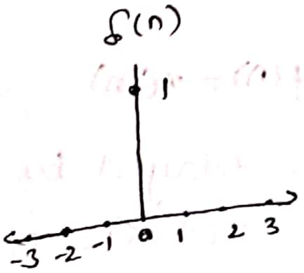
Representation of Arbitrary sequence:

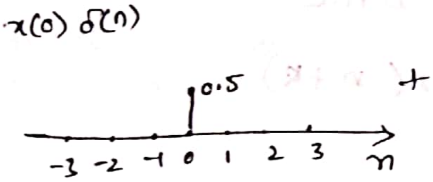
→ Any arbitrary sequence $x(n)$ can be represented interms of delayed and scaled impulse sequence $\delta(n)$.

Let $x(n)$ is infinite sequence. Then the sample $x(0)$ is obtained by multiplying $x(0)$, the magnitude, with unit-impulse $\delta(n)$.

i.e $x(0) \cdot \delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$







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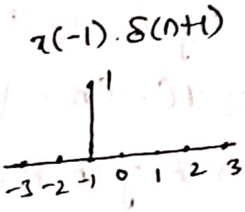

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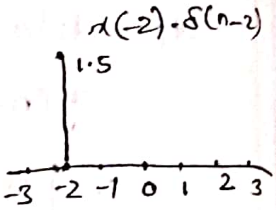

In The Same way if we represent The sequence

$$x(-1) \cdot \delta(n+1) = \begin{cases} x(-1) & \text{for } n=-1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

$$x(-2) \cdot \delta(n+2) = \begin{cases} x(-2) & \text{for } n=-2 \\ 0 & \text{for } n \neq -2 \end{cases}$$

$$x(0) \cdot \delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

$$x(1) \cdot \delta(n-1) = \begin{cases} x(1) & \text{for } n=1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$x(2) \cdot \delta(n-2) = \begin{cases} x(2) & \text{for } n=2 \\ 0 & \text{for } n \neq 2 \end{cases}$$

$$x(n) = \sum_{K=-\infty}^{\infty} x(K) \cdot \delta(n-K)$$

where $\delta(n-K)$ is unity for all $n=K$.

Impulse Response and Convolution:

$$y(n) = \sum_{K=-\infty}^{\infty} x(K) \cdot h(n-K)$$

$$= x(n) * h(n).$$

Properties of Convolution:

(i). Commutative law $x(n) * h(n) = h(n) * x(n)$

(ii) Associative law $[x(n) * h_1(n)] * h_2(n) \} = x(n) * [h_1(n) * h_2$

(iii) Distributive law $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x($

Causality:

Causal system is one whose output depends on Past or/and Present values of input.

using Convolution sum, we have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k).x(n-k)$$

$$= \underbrace{\sum_{k=-\infty}^{-1} h(k)\,x(n-k)}_{\text{futen i/ps}} + \underbrace{h(0).x(n)}_{\text{present i/p}} + \underbrace{\sum_{k=1}^{\infty} h(k).x(n-k)}_{\text{Past inputs}}.$$

from above Eq causal system h(k) is zero for k<0

(5) FIR and IIR systems:

LTI System can be classifide into Two types according to impulse response as IIR & FIR Systems.

FIR System: If the impulse response of The System is of finite duration, Then the system is called a FIR system.

ex: $h(n) = \begin{cases} 1 & \text{for } n=-1,2 \\ 2 & \text{for } n=1 \\ 3 & \text{for } n=0,3 \\ 0 & \text{otherwise} \end{cases}$

IIR system: impulse response is of infinite duration then system is IIR system

ex: $h(n) = a^n u(n)$

(6) Stable and unstable Systems:

Stability

An LTI System is stable if it produces a bounded output sequence for every bounded input sequence.

For a bounded input sequence x(n), The output is unbounded (infinite), Then system is unstable.

Let $x(n)$ - bounded input sequence
$h(n)$ - impulse response
$y(n)$ - output sequence

The Magnitude of output is $|y(n)| = \left|\sum_{K=-\infty}^{\infty} h(K) \cdot x(n-K)\right|$

We know that the magnitude of sum of terms is ≤ sum of magnitude., Hence

$$|y(n)| \leq \sum_{K=-\infty}^{\infty} |h(K)||x(n-K)|$$

Let bounded value of input is M,

$$\Rightarrow |y(n)| = M \cdot \sum_{K=-\infty}^{\infty} h(K)$$

The above condition will be satisfide when

$$\sum_{K=-\infty}^{\infty} |h(K)| < \infty$$

∴ The Necessary and Sufficient Condition for stability is $\boxed{\sum_{K=-\infty}^{\infty} |h(K)| < \infty}$

<u>Example:</u> Find The stability of the system whose impulse response $h(n) = (1/2)^n \cdot u(n)$

We know condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(K)| < \infty$$

given $h(n) = (1/2)^n \cdot u(n)$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |(1/2)^n \cdot u(n)| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1/2 + 1/2^2 + \ldots \ldots \infty \quad (\because 1+a+a^2 \ldots$$

$$= \frac{1}{1-1/2} = 2 < \infty$$

∴ System is stable.

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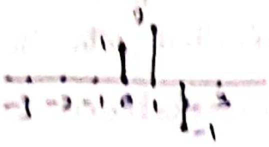
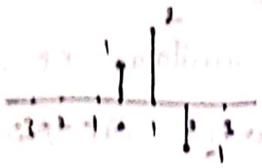
Example: Compute convolution of two sequences.

$x(n) = h(n) = \{\underset{\uparrow}{1}, 2, -1\}$

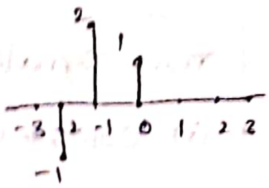
x(K)



h(K)



h(-K)



We know convolution b/w x(n) & h(n) is $y(n) = \sum_{K=-\infty}^{\infty} x(K)\cdot h(n-K)$

for n = 0, 1, 2

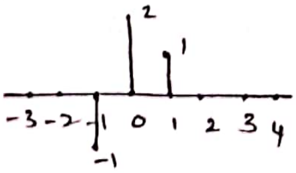
$y(0) = \sum_{K=-\infty}^{\infty} x(K)\cdot h(0-K)$

$= x(-2)\cdot h(-2) + x(-1)\cdot h(-1) + x(0)\cdot h(0) + x(1)\cdot h(1) + x(2)\cdot h(2)$

$= 0(-1) + 0(2) + 1(1) + 2(0) + (-1)(0) = 0 + 0 + 1 + 0 + 0$

$= 1$

h(1-K)



$y(1) = \sum_{K=-\infty}^{\infty} x(K)\cdot h(1-K)$

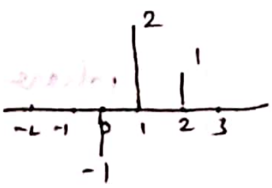
~~= x(-1)·h(~~ x(0)·h(1) ~~+~~ x(1)·h(0)

$= x(-1)\cdot h(-1) + x(0)\cdot h(0) + x(1)\cdot h(1) + x(2)\cdot h(2)$

$= 0(-1) + 1(2) + 2(1) + (-1)(0)$

$= 0 + 2 + 2 + 0 = 4$

h(2-K)



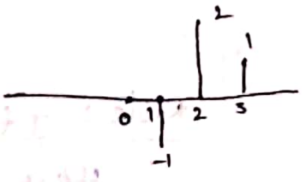
$y(2) = \sum_{K=-\infty}^{\infty} x(K)\cdot h(2-K)$

$= x(0)\cdot h(0) + x(1)\cdot h(1) + x(2)\, h(2)$

$= 1(-1) + 2(2) + (-1)+1$

$= -1 + 4 - 1 = 2$

h(3-K)

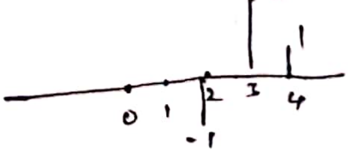


$y(3) = x(1)\cdot h(1) + x(2)\cdot h(2)\cdot x(3)\cdot h(3)$

$= 2(-1) + (-1)(2) + 0\cdot(1)$

$= -2 - 2 = -4$

h(4-K)



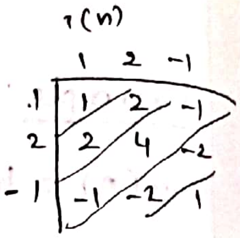
$y(4) = (-1)(-1) + 0(2) + 0(1)$

$= 1$

$\therefore \; y(n) = \{\underset{\uparrow}{1}, 4, 2, -4, 1\}$

$y(n) = \{1, 2+2, -1+4-1, -2-2, 1\}$

$= \{1, 4, 2, -4, 1\}$



Correlation: It is a Measure of degree to which Two signals are similar. It is divided into Two type

(i) Cross-Correlation. (ii) Auto correlation.

Cross correlation: It is measured between a pair of signals $x(n)$, $y(n)$ given by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n-l) \quad \text{for } l = 0, \pm1, \pm2, \pm3 \ldots$$

where l is shift (lag) parameter.

$$\gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n) \cdot x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l) \cdot x(n) \quad \text{——} ②$$

$$\gamma_{xy}(0) = \gamma_{yx}(0) = \sum_{n=-\infty}^{\infty} x(n) * y(n) \quad \text{——} ③$$

from Eq ① & ② $\boxed{\gamma_{xy}(l) = \gamma_{yx}(-l)}$

where $\gamma_{yx}(-l)$ is folded version of $\gamma_{xy}(l)$ at $l = 0$

$$\therefore \gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(-(l-n))$$

$$\boxed{\gamma_{xy}(l) = \cancel{\gamma_{y}} \cdot x(l) * y(-l)}$$

Note: It is computed similar to convolution But Here one sequence has been reversed

Auto-correlation: It is correlation of sequence with itself.

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\cdot x(n-l)$$

$$\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n+l)\cdot x(n)$$

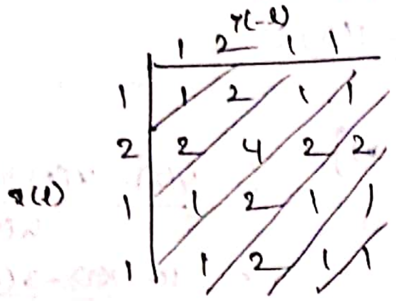
if $l=0$ → $\gamma_{xx}(l) = \sum_{n=-\infty}^{\infty} x^2(n)$

Example:

Find cross correlation of two finite length sequences

$x(n) = \{1,2,1,1\}$, $y(n) = \{1,1,2,1\}$

⇒ $y(-l) = \{1,2,1,1\}$, $x(l) = \{1,2,1,1\}$



$x(l)$ \ $y(-l)$	1	2	1	1
1	1	2	1	1
2	2	4	2	2
1	1	2	1	1
1	1	2	1	1

∴ $\gamma_{xy}(l) = \{1,4,6,6,5,2,1\}$

De-convolution: The process of recovering $x(n)$ from $x(n) * h(n)$ is known as deconvolution.

we know $y(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$

⇒ $y(0) = h(0)\cdot x(0)$

⇒ $y(1) = h(1)\cdot x(0) + h(0)\, x(1)$

⇒ $y(2) = h(2)\cdot x(0) + h(1)\cdot x(1) + h(0)\cdot x(2)$

⋮

⇒ $x(0) = \frac{y(0)}{h(0)}$, $x(1) = \frac{y(1) - h(1)\cdot x(0)}{h(0)}$, $x(2) = \frac{y(2) - h(2)\, x(0) - h(1)\cdot x(1)}{h(0)}$

So In general form,

$$\boxed{x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)\cdot h(n-k)}{h(0)}}$$

Example:

Find $x(n)$ if $y(n) = \{1, 5, 10, 11, 8, 4, 1\}$ and $h(n) = \{$

we know length of $y(n) = 7 = N_1 + N_2 - 1$

length of $h(n) = N_2 = 3$

$\Rightarrow$ length of $x(n) = N_1 = 7 + 1 - N_2$

$= 5$

We know $$x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k) \cdot h(n-k)}{y(0)}$$

$\therefore$ for $n = 0 \rightarrow$ $$x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1$$

$$x(1) = \frac{y(1) - x(0) \cdot h(1)}{h(0)} = \frac{5 - 1(2)}{1} = 3$$

$$x(2) = \frac{\left[y(2) - \sum_{k=0}^{1} x(k) \cdot h(2-k)\right]}{h(0)} = \frac{y(2) - x(0) \cdot h(2) - x(1)}{h(0)}$$

$$= \frac{10 - 1(1) - 3(2)}{1} = 3$$

$$x(3) = \frac{y(3) - \sum_{k=0}^{2} x(k) \cdot h(3-k)}{h(0)} = \frac{y(3) - x(0) \cdot h(3) - x(1) \cdot h(2) - x(}{h(0)}$$

$$= \frac{11 - 1(0) - 3(1) - 3(2)}{1} = 2$$

$$x(4) = \frac{y(4) - \sum_{k=0}^{3} x(k) \cdot h(4-k)}{h(0)} = \frac{y(4) - x(0) \cdot h(4) - x(1) \cdot h(3) - x(2) \cdot h(2) - x(3) \cdot}{h(0)}$$

$$= \frac{8 - 1(0) - 3(0) - 3(1) - 2(2)}{1} =$$

$\therefore$ $x(n) = \{1, 3, 3, 2, 1\}$

Examples:

① $x(n) = \{1, 2, -1\}$, $h(n) = \{1, 1, 1\}$

② $x(n) = u(n-1)$, $h(n) = (3)^n . u(-n-1)$

③ $x(n) = u(n)$, $h(n) = 5\,(-1/2)^n . u(n)$

④ $x(n) = \{3, 2, 1, 2\}$, $h(n) = \{1, 2, 1, 2\}$

⑤ $x(n) = \begin{cases} 1 & \text{for } n = -2, 0, 1 \\ 2 & n = -1 \\ 0 & \text{elsewhere} \end{cases}$ $h(n) = \delta(n) - \delta(n-1) + \delta(n-$

$- \delta$

Time Response Analysis of Discrete-time Systems

The general form of difference Equation of an N^{th} order linear time invariant discrete-time (LTI-DT) System is

$$\boxed{y(n) = -\sum_{K=1}^{N} a_K \cdot y(n-K) + \sum_{K=0}^{M} b_K \cdot x(n-K)}$$

where a_K & b_K are constants.

The response of any discrete-time System can be decomposed as

$$\boxed{\text{Total Response} = \text{zero state response} + \text{zero input response}}$$

→ zero state Response of the System due to i/p alone when initial state of System is zero. i.e System is initially relaxed at time $n=0$. [forced Response]

→ zero input Response depends only on the initial state of the System. i.e input is zero.

Let a 1st order Discrete-time System with difference Equation is

$$y(n) = a \cdot y(n-1) + x(n) \quad \text{where } x(n) - i/p, \; y(n) - o/p$$

for $n=0 \rightarrow$ $y(0) = a \cdot y(-1) + x(0)$

$n=1 \rightarrow$ $y(1) = a \cdot y(0) + x(1)$

$= a[a \cdot y(-1) + x(0)] + x(1)$

$= a^2 y(-1) + a \cdot x(0) + x(1)$

$n=2 \rightarrow$ $y(2) = a \cdot y(1) + x(2)$

$= a[a^2 y(-1) + a \cdot x(0) + x(1)] + x(2)$

$= a^3 y(-1) + a^2 x(0) + a \cdot x(1) + x(2)$

for any 'n'

$$y(n) = a^{n+1} y(-1) + a^n \cdot x(0) + a^{n-1} x(1) + \cdots + x(n)$$

$$\boxed{y(n) = a^{n+1} y(-1) + \sum_{K=0}^{n} a^K \cdot x(n-K) \quad \text{for } n \geq 0}$$

The response y(n) includes Two parts. The 1st part depends on initial Condition of the system. The 2nd term depends on i/p.

when $y(-1)=0$, The o/p response depends only on i/p which is known as zero state or forced response.

$$y_f(n) = \sum_{K=0}^{n} a^K . x(n-K) \quad \text{for } n \geq 0$$

If system is nonrelaxed i.e $y(-1) \neq 0$, and input $x(n)=0$, Then zero input response (or natural response)

$$y_n(n) = a^{n+1} y(-1) \quad \text{for } n \geq 0$$

The difference Eq of N^{Th} order discrete-time system is

$$\sum_{K=0}^{N} a_K . y(n-K) = \sum_{K=0}^{M} b_K . x(n-K). \quad \text{where } N\text{-order.}$$

The solution of difference Equation can be expressed sum of Two parts given by $\boxed{y(n) = y_h(n) + y_p(n)}$

$y_h(n)$ — Homogeneous or Complimentary solution

$y_p(n)$ — Particular solution.

Natural response or zero input response:

For a discrete-time system The natural response i solution of the Homogeneous Equation

$$\sum_{K=0}^{N} a_K y(n-K) = 0 \quad \text{—①}$$

The solution of Eq① is of The form $y_h(n) = \lambda^n$. —②

Substituting Eq② in ① $\Rightarrow \sum_{K=0}^{N} a_K . \lambda^{n-K} = 0$; $a_0 = 1$

or $\lambda^n + a_1 \lambda^{n-1} + \dots + a_{N-1} . \lambda^{n-(N-1)} + a_N . \lambda^{n-N} = 0$

$$\Rightarrow \lambda^{n-N} [\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N] = 0$$

which gives $\lambda^N + a_1 \lambda^{N-1} + \dots + a_{N-1} \lambda + a_N = 0$ —

The polynomial of Eq③ is called The characteristic Eq The system. Then N^{Th} order characteristic Eq can be in factorized form as $(\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_N) = 0$ -

where $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_N$ are roots of Eq ④ (3)

Eigen Values

The nature of Natural response depends on type of roots: real, imaginary and complex.

Distinct roots: If the roots $\lambda_1, \lambda_2 \cdots \lambda_N$ ~~of~~ are distinct, Then it has N solutions $C_1\lambda_1^n, C_2\lambda_2^n \cdots C_N \cdot \lambda_N^n$.

Then In general general solution is of the form

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N \cdot \lambda_N^n$$

where $C_1, C_2 \ldots C_N$ are arbitrary constants. These constants are determined by applying initial conditions.

Ex: if roots are $\lambda_1 = 2$, $\lambda_2 = 3$ Then $y_h(n) = C_1(2)^n + C_2(3)^n$

Repeated roots:

If the roots λ_1 is repeated m times and remaining (N−m) roots are distinct Then, characteristic Equation of the System is

$$(\lambda-\lambda_1)^m(\lambda-\lambda_{m+1})(\lambda-\lambda_{m+2})\cdots(\lambda-\lambda_N) = 0$$

general solution is

$$y_h(n) = (C_1+C_2n+C_3n^2+\cdots+C_m \cdot n^{m-1})(\lambda_1)^n + C_{m+1}(\lambda_{m+1})^n$$
$$+ C_{m+2}(\lambda_{m+2})^n + \cdots + C_N\lambda_N^n$$

Ex: If roots are $\lambda_1 = -2$, $\lambda_2 = -2$ and $\lambda_3 = 2$ Then

$$y_h(n) = [C_1 + C_2 n](-2)^n + C_3(2)^n$$

Complex roots:

If roots are complex, Then, we can write

$$\lambda_1 = \lambda = a + jb, \quad \lambda_2 = \lambda^* = a - jb$$

Then Homogeneous solution is of the form

$$y_h(n) = r^n[A_1 \cos n\theta + A_2 \sin n\theta]$$

where $r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}(b/a)$

A_1, A_2 are constants.

Example:

Find The natural response of The system describe differential Eq $y(n)+2y(n-1)+y(n-2) = x(n)+x(n-1)$ wi initial Conditions $y(-1) = y(-2) = 1$.

sol:

The Homogeneous Eq can be obtained by Equating i/p terms to zero. i.e.

$$y(n)+2y(n-1)+y(n-2) = 0 \quad -①$$

Homogeneous Solution is of The form $y_h(n) = \lambda^n$ -(

from ① & ②

$$\lambda^n + 2\cdot\lambda^{n-1} + \lambda^{n-2} = 0$$

$$\Rightarrow \lambda^{n-2}[\lambda^2 + 2\cdot\lambda + 1] = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 =$$

The roots are repeated, so The general form of Homogeneous solution is $y_h(n) = c_1(-1)^n + c_2 n(-1)^n$ —(

from Eq ③ $\quad y(0) = c_1$

$$y(1) = -c_1 - c_2$$

in Eq ① let $n = 0 \Rightarrow y(0) + 2y(-1) + y(-2) = 0$

$$y(0) + 2(1) + 10 = 0$$

$$\Rightarrow y(0) = -3$$

let $n = 1 \Rightarrow y(1) + 2y(0) + y(-1) = 0$

$$y(1) + 2(-3) + 1 = 0$$

$$\Rightarrow y(1) = 5$$

$\therefore c_1 = -3, \; c_2 = -2$

Hence Natural response is $y_n(n) = -3(-1)^n - 2n(-1)^n$ +

$$y_n(n) = -3(-1)^n \cdot u(n) - 2n(-1)$$

Forced Response (zero state Response) :-

It is solution of differential Equation for a given i/p when initial Conditions are zero. It contains Two parts
1. Homogoneous solution 2. Perticular Solution.

$$\boxed{y_f(n) = y_h(n) + y_p(n)}$$

The general form of perticular solution for several i/ps given by

$x(n)$ Input Signal	$y_p(n)$ Perticular solution
A (step input)	K
$A \cdot M^n$	$K \cdot M^n$
$A \cdot n^M$	$K_0 \cdot n^M + K_1 n^{M-1} + \ldots + K_M$
$A\cos\omega n$ $A\sin\omega n$	$c_1\cos\omega n + c_2\sin\omega n$

Example :

Find The forced Response of The System described by The difference Equation $y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$ for input $x(n) = (1/2)^n u(n)$

Sol:

In Previous Example, we find the $y_h(n) = c_1(-1)^n + c_2 n(-1)^n$ ——①

for i/p $x(n) = (1/2)^n u(n)$ The perticular solution is of The form

$$y_p(n) = K(1/2)^n u(n) \quad \text{——②}$$

Substituting $y_p(n)$, $x(n)$ in given difference Equation

$$K\left(\tfrac{1}{2}\right)^n + 2K(1/2)^{n-1} + K(1/2)^{n-2} = (1/2)^n + (1/2)^{n-1}$$

for $n = 2 \Rightarrow K(1/2)^2 + 2K(1/2) + K = 1/4 + 1/2$

$$\rightarrow \tfrac{1}{4}K + K + K = 3/4 \Rightarrow \boxed{K = 1/3}$$

$$\therefore \boxed{y_p(n) = \tfrac{1}{3}\left(\tfrac{1}{2}\right)^n} \quad \text{——③}$$

The forced response $y_f(n) = y_h(n) + y_p(n)$

$$y_f(n) = c_1(-1)^n + c_2 n(-1)^n + \tfrac{1}{3}\cdot\left(\tfrac{1}{2}\right)^n \quad \text{——④}$$

for $n=0 \Rightarrow y(0) = c_1 + 1/3$

for $n=1 \Rightarrow y(1) = -c_1 - c_2 + 1/6$ } — ⑤

from diff Eq $y(0) + 2y(-1) + y(-2) = x(0) + x(-1)$

for $n=0 \Rightarrow$

$y(0) = 1$ $[\because y(-1) = y(-2) = 0]$

for $n=1 \Rightarrow y(1) + 2y(0) + y(-1) = x(1) + x(0)$

$y(1) + 2(1) = 1 + 1/2$

$\Rightarrow y(1) = -1/2$ } — ⑥

from Eq ⑤ & ⑥

$1 = c_1 + 1/3 \Rightarrow c_1 = 1 - 1/3 = 2/3$

$-1/2 = -c_1 - c_2 + 1/6$

$\Rightarrow c_1 + c_2 = 1/6 + 1/2 \Rightarrow c_2 = 0$

$$\therefore \boxed{y_f(n) = \frac{2}{3}(-1)^n + \frac{1}{3}(1/2)^n}$$

Total Response :

It is obtained by adding natural response and forced response.

$$y(n) = y_h(n) + y_f(n)$$

* There is no need to find the forced response and natural response separately, The total res can be found in the same way as forced Respons by using actual initial conditions instead of initial conditions.

Exercises:

① Find natural response of system described by differ Eq $y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$ with $y(-1)=$

Ans: $y(n) = -2n(2)^n$

② Find forced response for above problem. with $x(n)=$

Ans: $\left(\frac{7}{9} + \frac{n}{3}\right)2^n u(n) + \frac{2}{9}(-1)^n$

Example: Find the response of the System described by the difference Equation $y(n)+2y(n-1)+y(n-2)=x(n)+x(n-1)$ for the input $x(n)=\left(\frac{1}{2}\right)^n u(n)$ with initial conditions $y(-1)=y(-2)=1$.

Sol:

The total response of the system is $y(n)=y_n(n)+y_f(n)$

In above Two Examples we Calculated $y_n(n)$ and $y_f(n)$.

$$y_n(n)=-3(-1)^n u(n)-2n(-1)^n u(n)$$

$$y_f(n)=\frac{2}{3}(-1)^n u(n)+\frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$\Rightarrow y(n)=y_n(n)+y_f(n)$$

$$\boxed{y(n)=-\frac{7}{3}(-1)^n u(n)-2n(-1)^n u(n)+1/3\left(1/2\right)^n u(n)}$$

Exercise: find total response of system described by diff. Eq $y(n)-4y(n-1)+4y(n-2)=x(n)-x(n-1)$ when i/p $x(n)=(-1)^n u(n)$ with initial conditions $y(-1)=y(-2)=1$.

Ans: $\left(\frac{7}{9}-\frac{5}{3}n\right)2^n u(n)+\frac{2}{9}(-1)^n u(n))$.

Impulse Response:

The N^{Th} order difference Equation is given by

$$\sum_{K=0}^{N} a_K y(n-K)=\sum_{K=0}^{M} b_K x(n-K) \quad N>M$$

$$\Rightarrow 1+\sum_{K=1}^{N} a_K y(n-K)=\sum_{K=0}^{M} b_K\cdot x(n-K) \quad N>M$$

for $x(n)=\delta(n)$, we obtain

$$1+\sum_{K=1}^{N} a_K y(n-K)=\sum_{K=0}^{M} b_K\delta(n-K) \quad \text{———} ①$$

for $n>M$ the above Equation reduces to Homogeneous Eq

$$\sum_{K=1}^{N} a_K\cdot y(n-K)=0 \;;\; a_0=1 \quad \text{———} ②$$

we can obtain $h(n)$ by solving Eq ② and imposing the initial Conditions to determine the arbitrary constants

if $N=M$, we have to add an impulse function to Homogeneous solution.

Example: Determine The impulse response $h(n)$ for system described by The difference Eq $y(n)=0.6y(n-1)-0.08y(n-2)+x(n)$.

Sol: Given $y(n)=0.6y(n-1)-0.08y(n-2)+x(n)$ ——①

we know total Response $y(n)=y_h(n)+y_p(n)$

for impulse $x(n)=\delta(n)$ The particular solution:

$$y_p(n)=0 \rightarrow y(n)=y_h(n)$$

Homogeneous solution is obtained by $x(n)=0$

$$\Rightarrow y(n)-0.6y(n-1)+0.08y(n-2)=0 \quad \text{——②}$$

let solution $y_h(n)=\lambda^n$ ——③

$\Rightarrow$ from Eq② & ③

$$\lambda^n-0.6\lambda^{n-1}+0.08\lambda^{n-2}=0$$

$$\Rightarrow \lambda^{n-2}[\lambda^2-0.6\lambda+0.08]=0$$

The roots of characteristic Eq are

$$\Rightarrow \lambda_1=0.4,\ \lambda_2=0.2$$

The general solution of Homogeneous Eq is

$$y_h(n)=c_1\lambda_1^n+c_2\lambda_2^n$$
$$=c_1[0.4]^n+c_2[0.2]^n$$

$$\left.\begin{aligned} \text{at } n=0 &\Rightarrow y(0)=c_1+c_2 \\ n=1 &\rightarrow y(1)=0.4c_1+0.2c_2 \end{aligned}\right\} \text{——④}$$

from Eq①

$$y(0)=0.6y(-1)-0.08y(-2)+x(0)$$
$$=1 \quad \left[\because \begin{aligned} y(-1)=y(-2)=0 \\ x(0)=\delta(0)=1 \end{aligned}\right]$$

$$y(1)=0.6y(0)-0.08y(-1)+x(1)$$
$$=0.6(1)-0.08(0)+0$$
$$=0.6$$

from Eq ④ & ⑤ $\Rightarrow c_1+c_2=1,\ 0.4c_1+c_2 0.2=0.6$

$$\Rightarrow c_1=2,\ c_2=-1$$

$$\therefore \boxed{y(n)=2(0.4)^n u(n)-(0.2)^n u(n)}$$

Example-2: Determine the impulse response $h(n)$ for the system described by difference Eq

$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$.

sol: given $\quad y(n)+y(n-1)-2y(n-2) = x(n-1)+2x(n-2)$ —①

Here $N=M=2$, so Homogeneous solution includes impulse term.

The response is given by $y(n) = y_h(n) + y_p(n)$.

for the input $x(n) = \delta(n)$, The particular solution $y_p(n) = 0$.

$$\Rightarrow y(n) = y_h(n)$$

The Homogeneous Solution can be found by Equating i/p terms to zero. i.e

$$y(n) + y(n-1) - 2y(n-2) = 0$$

solution is $\quad y_h(n) = \lambda^n$

Then

$$\lambda^n + \lambda^{n-1} - 2\lambda^{n-2} = 0$$

$$\Rightarrow \lambda^{n-2}[\lambda^2+\lambda-2]=0 \Rightarrow \lambda^2+\lambda+2=0$$

$$\lambda_1 = 1, \lambda_2 = -2$$

Then general form is $\quad y_h(n) = C_1(1)^n + C_2(-2)^n + A.\delta(n)$ —②

$$\left.\begin{aligned} &\text{for } n=0 \rightarrow y(0) = C_1+C_2+A \\ &n=1 \rightarrow y(1) = C_1 - 2C_2 \\ &n=2 \Rightarrow y(2) = C_1 + 4C_2 \end{aligned}\right\} \text{—③}$$

from difference Equation

$n=0 \Rightarrow \quad y(0)+y(-1)-2y(-2) = x(-1)+2x(-2)$

$\qquad y(0) = 0$

$n=1 \Rightarrow \quad y(1)+y(0)-2y(-1) = x(0)+2x(-1)$

$\qquad y(1)+0-2(0) = 1+2(0)$

$\qquad y(1) = 1$

$n=2 \Rightarrow \quad y(2)+y(1)-2y(0) = x(1)+2x(0)$

$\rightarrow \quad y(2)+1-2(0) = 0+2$

$\rightarrow \quad y(2) = 1$

④

from Eq ⑤ & ④

$c_1 + c_2 + A = 0$ ——— 5(a)

$c_1 - 2c_2 = 1$ ——— 5(b)

$c_1 + 4c_2 = 1$ ——— 5(c)

from 5(b) & 5(c)

5(b) − 5(c)

$$\begin{aligned} c_1 - 2c_2 &= 1 \\ -c_1 + 4c_2 &= -1 \\ \hline 2c_2 &= 0 \end{aligned}$$

$\Rightarrow c_2 = 0$

$\Rightarrow c_1 = 1$

from 5(a)

$1 + 0 + A = 0 \Rightarrow A = -1$

$$\therefore \; y(n) = (1)^n u(n) + (-1).\delta(n)$$

$$= u(n) - \delta(n)$$

$$\boxed{y(n) = u(n-1)}$$

Exercise: Determine impulse response $h(n)$ for system descri
by $y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$

Ans: $\frac{n}{2}(2)^n$

Step response:

The step response can be easily expressed interms of
impulse response using convolution sum. Let a discr
-time system have impulse response as $s(n)$.

Then $s(n) = h(n) * u(n)$

$$= \sum_{K=-\infty}^{\infty} h(K).u(n-K)$$

Since $u(n-K) = 0$ for $K > n$ and $u(n-K) = 1$ for $K \le n$, we have

$$s(n) = \sum_{K=-\infty}^{\infty} h(K).$$

i.e step response is running sum of impulse response

Example:

Find The impulse response and step response of a discrete-time LTI System whose difference Equation is

$y(n) = y(n-1) + 0.5\,y(n-2) + x(n) + x(n-1)$.

Sol: given Eq $y(n) = y(n-1) + 0.5\,y(n-2) + x(n) + x(n-1)$

for impulse response, particular solution $y_p(n) = 0$

$$\therefore \; y(n) = y_h(n)$$

The Homogeneous sol is $y(n) = \lambda^n$

$\Rightarrow \lambda^n - \lambda^{n-1} - 0.5\lambda^{n-2} = 0 \quad [\because x(n) = 0]$

$\rightarrow \lambda^{n-2}[\lambda^2 - \lambda - 0.5] = 0 \rightarrow \lambda^2 - \lambda - 0.5 = 0 \rightarrow \lambda_1 = 1.266$

$\lambda_2 = -0.366$

$\therefore y_h(n) = c_1(1.366)^n + c_2(-0.366)^n$

for $n = 0 \rightarrow y(0) = c_1 + c_2$

$n = 1 \Rightarrow y(1) = 1.366c_1 - 0.366c_2$

from diff Eq $y(0) = 1$

$y(1) = 2$

$\therefore c_1 = 1.366, \; c_2 = -0.366$

$\therefore y(n) = 1.366(1.366)^n - 0.366(-0.366)^n$.

<u>step response</u>:

for input $x(n) = u(n)$, $y_p(n) = k \cdot u(n)$ Then substituting $x(n), y_p(n)$ in differencial Equation.

$$k \cdot u(n) = k \cdot u(n-1) + 0.5k\,u(n-2) + u(n) + u(n-1)$$

for $n = 2$, where none of the terms vanish we get

$$k = k + 0.5k + 1 + 1$$

$$\Rightarrow k = -4$$

$$\therefore y_p(n) = -4u(n)$$

Total response $y(n) = y_h(n) + y_p(n)$

$= c_1(1.366)^n + c_2(-0.366)^n - 4u(n)$

for $n = 0 \Rightarrow y(0) = c_1 + c_2 - 4$

$y(1) = 1.366c_1 - 0.366c_2 - 4$

from diff eq $y(0) = 1$, $y(1) = 4$.

$\Rightarrow c_1 = 5.098, \; c_2 = -0.098$

$\therefore$ step response $y(n) = 5.098(1.366)^n - 0.098(-0.366)^n - 4$ for $n \geq 0$

$\Rightarrow y(n) = 5.098(1.366)^n u(n) - 0.098(-0.366)^n u(n) - 4u(n)$

<u>Exercise</u>: ① find impulse & step response for $y(n) + y(n-1) = x(n) - 2x(n-1)$

<u>Ans</u>: $3(-1)^n u(n) - 2\delta(n)$

$5/2(-1)^n u(n) - 1/2\, u(n)$

② $y(n) - y(n-1) - y(n-2) = x(n-1)$, system initially relaxed then impulse

<u>Ans</u>: $y(n) = 0.447(1.618)^n u(n) - 0.447(-0.618)^n u(n)$

③ solution find for $y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$ for $x(n) = 2^n u(n)$

Time Response Analysis of Discrete-time System

Exercises:

① Determine impulse response $h(n)$ for system described by The second order difference Equations

$$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$$

Ans: $y(n) = 2(0.4)^n u(n)$.

② $y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

[for N=M so response = $y_h(n) + A \cdot \delta(n)$]

Ans: $y(n) = u(n) - \delta($
$= u(n-1)$

③ $y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$

Ans: $y(n) = \frac{n}{2}(2)^n$.

④ find impulse and step response of system with dif

$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$

~~step~~ impulse **Ans:** $y(n) = 1.366(1.366)^n - 0.36$

step **Ans:** $y(n) = 5.098(1.366)^n - 0.098($

⑤ find impulse & step response of system

$y(n) + y(n-1) = x(n) - 2x(n-1)$

Ans: $3(-1)^n u(n) - 2\delta($
$3/2(-1)^n u(n) - 1/2$

⑥ find solution of difference Eq

$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$ for $x(n) = 2^n u(n)$.

Frequency Analysis of Discrete-time ~~Systems~~ signals

Any continuous-time periodic signal $x(t)$ with period T can be represented as a weighted sum of Harmonically related sinusoids or complex exponentials.

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)] \quad ①$$

where $a_0, a_1, a_2 \cdots a_n$, $b_1, b_2 \cdots b_n$ are called as fourier coefficients and Ω_0 is fundamental frequency equal to $\frac{2\pi}{T}$

Then the exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\Omega_0 t} \quad \text{where } \Omega_0 = \frac{2\pi}{T} \quad ②$$

when we represent in discrete-time signal.

$$x(n) = \sum_{K=0}^{N-1} C_K \cdot e^{jK(\frac{2\pi}{N})n} \quad ③$$

to find fourier series coefficients C_K multiply Eq ③ both sides by $\sum_{n=0}^{N-1} e^{-jm(\frac{2\pi}{N})n}$

$$\Rightarrow \sum_{n=0}^{N-1} x(n) \cdot e^{-jm(\frac{2\pi}{N})n} = \sum_{n=0}^{N-1}\left[\sum_{K=0}^{N-1} C_K e^{jK(\frac{2\pi}{N})n}\right] \cdot e^{-jm(\frac{2\pi}{N})n}$$

$$= \sum_{K=0}^{N-1} C_K \cdot \sum_{n=0}^{N-1} e^{j(K-m)(\frac{2\pi}{N})n}$$

using the relation

$$\sum_{n=0}^{N-1} e^{j(\frac{2\pi}{N})(k-m)n} = N \quad \text{if } k-m = 0, \pm N, \pm 2N \ldots$$

$$= 0 \quad \text{otherwise}$$

we obtain

$$\sum_{n=0}^{N-1} x(n) \cdot e^{-jm(\frac{2\pi}{N})n} = N \cdot C_m$$

(or)

$$C_m = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-jm(\frac{2\pi}{N})n}$$

$$C_K = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} \quad \text{where } k = 0, 1, 2 \ldots N-1$$

$$\Rightarrow \boxed{\begin{array}{l} C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi nk}{N}} \quad \text{where } k=0,1,2\cdots N-1 \\ x(n) = \sum_{k=0}^{N-1} C_k\cdot e^{j\frac{2\pi nk}{N}} \quad \text{where } n=0,1,2\cdots N-1 \end{array}}$$

C_k is periodic with period N

$$C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot e^{-j\frac{2\pi}{N}\cdot n\,(k+N)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot e^{-jk\left(\frac{2\pi}{N}\right)n}\cdot e^{-jN\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{C_{k+N} = C_k.}$$

Discrete-Frequency Spectrum & frequency Range

→ Consider a periodic sequence $x(n)$ with period N. This sequence can be expressed in discrete Fourie Series as $x(n) = \sum_{k=0}^{N-1} C_k\cdot e^{\frac{j2\pi kn}{N}}$

→ The values of C_k, $k=0,1,2\cdots N-1$ are called the discr line spectra of $x(n)$. Each C_k appears at freq ω_k=

The interval is $0 \le \omega < 2\pi$ or $-\pi < \omega \le \pi$,

→ The Fourier coefficients C_k are complex and they be represented in polar form as $C_k = |C_k|\cdot e^{j\angle C_k}$

→ the plot of $|C_k|$ Vs ω is called magnitude sp The plot of $\angle C_k$ Vs ω is called phase Spectru

Properties of Discrete-time Fourier Series:

Property	Periodic Signal	Fourier Series Coefficients
~~Linearity~~	$x(n)$ with period N $y(n)$ with period N $w_0 = \frac{2\pi}{N}$	C_K with period N D_K with period N
Linearity	$a_1 x(n) + a_2 y(n)$	$a_1 C_K + a_2 D_K$
Time shifting	$x(n-n_0)$	$C_K \cdot e^{-j\frac{2\pi K n_0}{N}}$
Frequency shifting	$e^{j\frac{2\pi M n}{N}} \cdot x(n)$	C_{K-M}
Time Reversal	$x(-n)$	C_{-K}
conjugation	$x^*(n)$	C^*_{-K}
convolution	$\sum_{r=\langle N\rangle} x(r) \cdot y(n-r)$	$N \cdot C_K D_K$
Multiplication	$x(n) \cdot y(n)$	$\sum_{l=\langle N\rangle} C_l D_{K-l}$
Conjugate symmetry for real signals	$x(n)$ real	$C_K = C^*_{-K}$ $\text{Re}\{C_K\} = \text{Re}\{C_{-K}\}$ $\text{Im}\{C_K\} = -\text{Im}\{C_K\}$ $\lvert C_K\rvert = \lvert C_{-K}\rvert$ $\angle C_K = -\angle C_{-K}$
Real and Even signals	$x(n)$ real and Even	C_K real and Even
Real and odd signals	$x(n)$ real and odd	C_K imaginary and odd
	$x_e(n)$	$\text{Re}\{C_K\}$
	$x_o(n)$	$j\text{Im}\{C_K\}$

Parseval's relation for Periodic signals.

$$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x(n)\rvert^2 = \sum_{K=\langle N\rangle} \lvert C_K\rvert^2.$$

Discrete-time Fourier Transform

Let us consider a periodic sequence $x_N(n)$ with peri
as $N \to \infty$ the periodic sequence $x_N(n)$ becomes aper
sequence $x(n)$.

i.e $x(n) = \lim_{N\to\infty} x_N(n)$.

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_N(n) e^{-j\frac{2\pi k n}{N}} \quad \text{———} ①$$

where $$x_N(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi k n}{N}} \quad \text{———} ②$$

for $N \to \infty$, The coefficients C_K will tend to zero.

Hence we define the new fourier series Coeffici
as

$$NC_K = X(e^{j\frac{2\pi K}{N}}) \quad \text{———} ③$$

let $\omega_K = \frac{2\pi}{N}K$ Then $NC_K = X(e^{j\omega_K})$

let $x_N(n)$ is periodic with period N, Then Eq① c
be ~~worri~~ taken from $n = -(\frac{N}{2}-1)$ to $\frac{N}{2}$

$$\Rightarrow C_K = \frac{1}{N}\sum_{n=-(\frac{N}{2}-1)}^{N/2} x_N(n) e^{-j\frac{2\pi k n}{N}}$$

$$\Longrightarrow NC_K = \sum_{n=-(\frac{N}{2}-1)}^{N/2} x_N(n) \cdot e^{-j\frac{2\pi K n}{N}} \quad \text{———} ④$$

from Eq ③ & ④

$$X(e^{j\frac{2\pi K}{N}}) = X(e^{j\omega_K}) = \sum_{n=-(\frac{N}{2}-1)}^{N/2} x_N(n) e^{j\omega_K n}$$

as $N \to \infty$, the periodic sequence becomes aperiodic
and $\frac{2\pi}{N} \to 0$, In cas $\omega_K = \frac{2\pi}{N}K$ becomes a continuous var
ω and limits becomes $-\infty$ to ∞

$$\therefore \boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}} \quad \text{———} ⑤$$

we know

$$x_N(n) = \sum_{n=0}^{N-1} C_K e^{j\frac{2\pi Kn}{N}}$$

$$= \frac{1}{2\pi} \sum_{n=0}^{N-1} N C_K . e^{j\frac{2\pi Kn}{N}} \left(\frac{2\pi}{N}\right) \quad \text{——} ⑥$$

$N \to \infty$, $\frac{2\pi}{N} \to 0$ and $\omega = \frac{2\pi}{N} K$ becomes continuous and $\frac{2\pi}{N}$ can be written as $d\omega$.

Then Σ in Eq⑥ becomes integral with limits $0 \to 2\pi$.

$$\therefore \quad x(n) = \frac{1}{2\pi} \int_{0}^{2\pi} X(e^{j\omega}) . e^{j\omega n} \, d\omega$$

$$\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \, e^{j\omega n} \, d\omega} \quad \text{——} ⑦$$

Eq ⑤ & ⑦ are DTFT pair.

$\Rightarrow \quad X(e^{j\omega}) = F[x(n)]$

$\Rightarrow \quad x(n) = F^{-1}[X(e^{j\omega})]$

Note: DTFT does not exist for every aperiodic sequence. The sufficient condition for existency is

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

i.e $x(n)$ absolutely summable.

Examples:

Find Fourier Transform of following

① $\delta(n)$

given $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \left[\begin{array}{l} \delta(n) = 0 \text{ for } n \neq 0 \\ \quad\;\; = 1 \text{ for } n = 0 \end{array}\right]$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) e^{j\omega n} = 1$$

$F[\delta(n)] = 1$

(ii) $x(n) = u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega} + \dots \infty$$

$[\because 1 + r + r^2 \dots \infty =$

$$F(u(n)) = \frac{1}{1-e^{-j\omega}}$$

(iii) $x(n) = \delta(n-K)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \delta(n-K) \cdot e^{-j\omega n}$$

$$= e^{-j\omega K}$$

(iv) $x(n) = u(n-K)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u(n-K) e^{-j\omega n}$$

$$= \sum_{n=K}^{\infty} e^{-j\omega n} = e^{-j\omega K} + e^{-j\omega(K+1)} + \dots$$

$$= e^{-j\omega K}\left[1 + e^{-j\omega} + \dots - \right]$$

$$= \frac{e^{-j\omega K}}{1-e^{-j\omega}}$$

② Find The DTFS for $x(n) = \cos \pi/3\, n$

Sol: fundamental period of $x(n)$ is $N = \frac{2\pi}{\omega_0}(m)$

$$\Rightarrow \frac{2\pi}{\pi/3} m = 6 \text{ (for } m=1)$$

the Discrete-time signal $x(n)$ can be expressed as

$$x(n) = \frac{1}{2}\left[e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right] = \frac{1}{2} e^{j\frac{\pi}{3}n} + \frac{1}{2} e^{-j\frac{\pi}{3}n} \quad \text{—①}$$

we know $x(n) = \sum_{(N)} c_k e^{j\frac{2\pi k n}{N}}$

at $K = -2$ to 3 $\quad x(n) = \sum_{K=-2}^{3} c_K e^{j\frac{2\pi k n}{N}}$ for

$$= \sum_{K=-2}^{3} C_K e^{j\frac{\pi}{3} nK} \quad \text{—②}$$

Equating The terms from Eq① & ②

$$C_K = \frac{1}{2} \text{ for } K = -1 \text{ and } 1$$

$$= 0 \text{ for } -2 \le K \le 3,\ K \ne -1 \text{ and } K \ne 1$$

Properties of Discrete-time Fourier Transform:

ⓐ Linearity

If $F[x_1(n)] = X_1(e^{j\omega})$ and $F[x_2(n)] = X_2(e^{j\omega})$ Then

$$F[a_1 x_1(n) + a_2 x_2(n)] = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Proof:

$$F[a_1 x_1(n) + a_2 x_2(n)] = a_1 F[x_1(n)] + a_2 f[x_2(n)]$$
$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

ⓑ Periodicity:

The DTFT $X(e^{j\omega})$ is periodic in ω with period 2π

$$X(e^{j\omega}) = X[e^{j(\omega + 2\pi k)}]$$ for any integer k (ω belongs to $[0, 2\pi]$ a $[-\pi, \pi]$)

ⓒ Time shifting

If $F[x(n)] = X(e^{j\omega})$ Then $F[x(n-k)] = e^{-j\omega k} \cdot X(e^{j\omega})$ where k-integer.

Proof:

$$F[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) \cdot e^{-j\omega n}$$

let $n-k = p$
$\Rightarrow n = p+k$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot e^{-j\omega(p+k)}$$

$$= e^{-j\omega k} \sum_{p=-\infty}^{\infty} x(p) e^{-j\omega p} = e^{-j\omega k} \cdot X(e^{j\omega})$$

* It doest change Amplitude spectrum, the phase spectrum is changed by $-\omega k$.

ⓓ Frequency shifting

If $F[x(n)] = X(e^{j\omega})$ Then $F[x(n) e^{j\omega_0 n}] = X[e^{j(\omega - \omega_0)}]$

it is dual of time shifting property.

Proof:

$$F[x(n) e^{j\omega_0 n}] = \sum_{n=-\infty}^{\infty} x(n) e^{j\omega_0 n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega - \omega_0)n}$$

$$= X[e^{j(\omega - \omega_0)}]$$

ⓔ Time reversal:

If $F[x(n)] = X(e^{j\omega})$ Then $F[x(-n)] = X(e^{-j\omega})$.

Proof:

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n).e^{j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-(-j\omega)n}$$

$$= X(e^{-j\omega})$$

Folding in Time domain corresponding to folding in frequency domain.

ⓕ Differentiation in frequency:

If $F[x(n)] = X(e^{j\omega})$ Then $F[nx(n)] = j\frac{d}{d\omega}(X(e^{j\omega}))$

Proof:

$$X(e^{j\omega}) = F[x(n)] = \sum_{n=-\infty}^{\infty} x(n).e^{-j\omega n}$$

differentiate both sides with respect to ω

$$\Rightarrow \frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} (-jn)\cdot x(n) e^{-j\omega n}$$

$$= (-j) \sum_{n=-\infty}^{\infty} n\cdot x(n)\cdot e^{-j\omega n}$$

$$\Rightarrow j\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} n\cdot x(n) e^{-j\omega n}$$

$$= F[n\cdot x(n)]$$

Therefore $F[n\cdot x(n)] = j\frac{dX(e^{j\omega})}{d\omega}$.

ⓖ Time convolution:

If $F[x_1(n)] = X_1(e^{j\omega})$ and $F[x_2(n)] = X_2(e^{j\omega})$ then

$F[x_1(n) * x_2(n)] = X_1(e^{j\omega}).X_2(e^{j\omega})$

Proof:

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$$

$$F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k).e^{-j\omega n}$$

Interchanging The order of Summation we get

$$F[x_1(n) * x_2(n)] = \sum_{k=-\infty}^{\infty} x(k) \cdot \sum_{n=-\infty}^{\infty} h(n-k) \cdot e^{-j\omega n}$$

let $n-k = P \Rightarrow F[x_1(n) * x_2(n)] = \sum_{k=-\infty}^{\infty} x(k) \sum_{P=-\infty}^{\infty} h(P) e^{-j\omega P} \cdot e^{-j\omega k}$

$$= \sum_{k=-\infty}^{\infty} x(k) . H(e^{j\omega}) . e^{-j\omega k}$$

$$= H(e^{j\omega}) \sum_{k=-\infty}^{\infty} x(k) \cdot e^{-j\omega k}$$

$$= H(e^{j\omega}) . X(e^{j\omega})$$

Convolution of two signals in Time domain is Equal to multiplying their spectra in frequency domain.

(h) Frequency convolution:

$$F[x_1(n) * x_2(n)] = X_1(e^{j\omega}) * X_2(e^{j\omega})$$

$$= \frac{1}{2\pi} \int_{2\pi}^{2\pi} X_1(e^{j\theta}) . X_2(e^{j(\omega-\theta)}) . d\theta$$

Proof:

$$F[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n) . e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_2(n) e^{-j\omega n} \left[\frac{1}{2\pi} \int_0^{2\pi} X_1(e^{j\theta}) . e^{j\theta n} d\theta \right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} X_1(e^{j\theta}) \left\{ \sum_{n=-\infty}^{\infty} x_2(n) e^{-j(\omega-\theta)n} \right\} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} X_1(e^{j\theta}) . X_2(e^{j(\omega-\theta)}) . d\theta .$$

(i) Symmetry property:

$x^*(n) \longleftrightarrow X^*(e^{-j\omega})$

$x^*(-n) \longleftrightarrow X^*(e^{j\omega})$

$x_R(n) \longleftrightarrow X_e(e^{j\omega})$

$x_I(n) \longleftrightarrow X_o(e^{j\omega})$

$x_e(n) \longleftrightarrow X_R(e^{j\omega})$

$x_o(n) \longleftrightarrow jX_I(e^{j\omega})$

(j) Parseval's Theorem:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

Proof:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\cdot x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cdot e^{j\omega n}\right]^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right]\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\cdot X(e^{j\omega})\cdot d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \cdot d\omega$$

(k) Correlation Theorem

$F[x_1(n)] = X_1(e^{j\omega})$ and $F[x_2(n)] = X_2(e^{j\omega})$ then

$F[\gamma_{x_1x_2}(l)] = \Gamma_{x_1x_2}(e^{j\omega}) = X_1(e^{j\omega})\cdot X_2(e^{-j\omega})$

(l) Modulation Theorem

$F[x(n)] = X(e^{j\omega})$ then $F[x(n)\cos\omega_0 n] = \frac{1}{2}\{X(e^{j(\omega+\omega_0)}\} + X(e^{j(\omega-$

Proof:

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n}\cdot\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$$

$$= \frac{1}{2}\left[X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)})\right]$$

(m) Wiener Khintchine Theorem.

If $x(n)$ is random signal, then the F.T of auto-cor sequence is the energy density Spectrum.

$$F[\gamma_{xx}(l)] = \Gamma_{xx}(e^{j\omega}).$$

Symmetry properties:

The Fourier Transform $X(e^{j\omega})$ is a Complex function of ω and can be expressed as $X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$.

↳ real ↳ Imaginary

Proof:

we know $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} x(n)[\cos\omega n - j\sin\omega n]$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cos\omega n - j\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

$$\Rightarrow X_R(e^{j\omega}) + jX_I(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n - j\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

Comparing LHS & RHS

$$X_R(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n$$

$$X_I(e^{j\omega}) = -\sum_{n=-\infty}^{\infty} x(n)\sin\omega n$$

Since $\cos(-\omega)n = \cos\omega n$, $\sin(-\omega)n = -\sin\omega n$,

$$X_R(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cos(-\omega)n = \sum_{n=-\infty}^{\infty} x(n)\cos\omega n = X_R(e^{j\omega}) \text{ (Even Symmetry)}$$

Similarly $X_I(e^{-j\omega}) = -X_I(e^{j\omega})$ (odd symmetry)

If we represent $X(e^{j\omega})$ in polar form $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$

$$X(e^{j\omega}) = |X(e^{j\omega})|[\cos\theta(\omega) + j\sin\theta(\omega)].$$

$$\Rightarrow X_R(e^{j\omega}) = |X(e^{j\omega})|\cos\theta(\omega), \quad X_I(e^{j\omega}) = |X(e^{j\omega})|\sin\theta(\omega)$$

$$\Rightarrow |X(e^{j\omega})| = \sqrt{[X_R(e^{j\omega})]^2 + [X_I(e^{j\omega})]^2}$$

$$\theta(\omega) = \tan^{-1}\left[\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right]$$

Similarly $|X(e^{-j\omega})| = |X(e^{j\omega})|$

$$\theta(\omega) = -\tan^{-1}\left(\frac{X_I(e^{j\omega})}{X_R(e^{j\omega})}\right).$$

Frequency Response Analysis of Discrete- time systems.

The o/p y(n) of any linear time invariant system to an i/p x(n) can be obtained using Convolution sum

$$y(n) = \sum_{K=-\infty}^{\infty} h(K)x(n-K)$$ where h(K) impulse Response of System.

let complex exponential Signal $x(n) = e^{j\omega n}$ as i/p

Then o/p is
$$y(n) = \sum_{K=-\infty}^{\infty} h(K).e^{j\omega(n-K)}$$
$$= e^{j\omega n} \sum_{K=-\infty}^{\infty} h(K).e^{-j\omega K}$$
$$= \underset{\substack{\downarrow \\ i/p}}{e^{j\omega n}}.\underset{\substack{\downarrow \\ \text{frequency response}}}{[H(e^{j\omega})]}$$

$$\Rightarrow \boxed{H(e^{j\omega}) = \sum_{K=-\infty}^{\infty} h(K).e^{-j\omega n}}$$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$$
$$\theta(\omega) = \angle H(e^{j\omega})$$

Frequency Response of 1st order System

The difference Equation of 1st order System is
$$y(n) - ay(n-1) = x(n)$$

Applying FT. $Y(e^{j\omega}) - a.e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega})$

$$\Rightarrow Y(e^{j\omega})[1 - ae^{-j\omega}] = X(e^{j\omega})$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{(1-ae^{-j\omega})}$$

The impulse Response $h(n) = F^{-1}[H(e^{j\omega})]$
$$= F^{-1}\left[\frac{1}{1-ae^{-j\omega}}\right]$$
$$= a^n u(n)$$

∴ Frequency response of 1st order system is

$$H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$= \frac{1}{1-a\cos\omega-aj\sin\omega}$$

magnitude response is

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2+a^2\sin^2\omega}}$$

$$= \frac{1}{\sqrt{1+a^2-2a\cos\omega}}$$

phase Response is $\angle H(e^{j\omega}) = -\tan^{-1}\frac{a\sin\omega}{1-a\cos\omega}$

Example:

Determine and sketch magnitude and phase response of $y(n) = \frac{1}{2}[x(n)+x(n-2)]$

Sol: given $y(n) = \frac{1}{2}[x(n)+x(n-2)]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\,e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}[x(n)+x(n-2)]e^{-j\omega n}$$

$$= \frac{1}{2}\left[\sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} + \sum_{n=-\infty}^{\infty} x(n-2)\,e^{-j\omega n}\right]$$

$$= \frac{1}{2}\left[X(e^{j\omega}) + e^{-2j\omega}X(e^{j\omega})\right]$$

$$= \cancel{X(e^{j\omega})}\ \frac{X(e^{j\omega})}{2}\left[1+e^{-2j\omega}\right]$$

$$\Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+e^{-2j\omega}}{2} = \frac{1+\cos 2\omega - j\sin 2\omega}{2}$$

$$\therefore |H(e^{j\omega})| = \frac{1}{2}\sqrt{(1+\cos 2\omega)^2+\sin^2 2\omega} = \frac{1}{2}\sqrt{2(1+\cos 2\omega)} = \frac{1}{2}\sqrt{2\cdot 2\cos^2\omega}$$

$$= \frac{1}{2}\cdot 2\cos\omega = \cos\omega$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-\sin 2\omega}{1+\cos 2\omega}\right)$$

$$= \tan^{-1}\left(\frac{-2\sin\omega\cos\omega}{\cancel{1}\,2\cos^2\omega}\right)$$

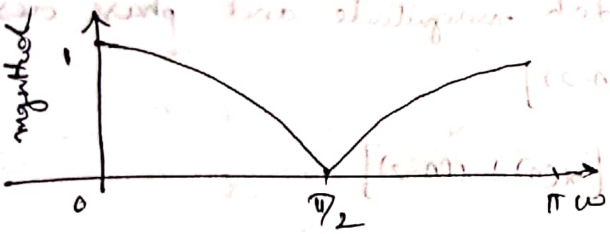
$$= \tan^{-1}(-\tan\omega) = -\omega$$

$$\therefore \angle H(e^{j\omega}) = -\omega \text{ for } H(e^{j\omega}) \geq 0$$

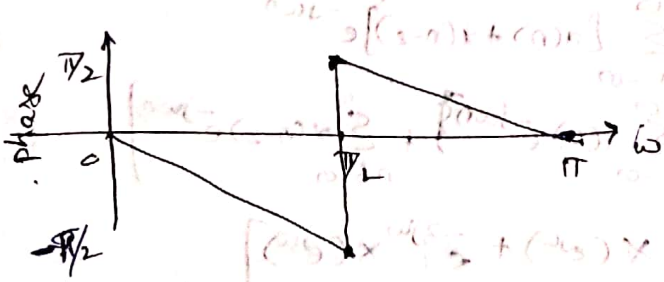
$$= -\omega+\pi \text{ for } H(e^{j\omega}) \leq 0$$

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$H(e^{j\omega})$	1	0.812	0.707	0.5	0	-0.5	-0.707	-1
$\lvert H(e^{j\omega})\rvert$	1	0.812	0.707	0.5	0	0.5	0.701	1
$\angle H(e^{j\omega})$	0	$-\pi/6$	$-\pi/4$	$-\pi/3$	$-\pi/2$	$\pi/3$	$\pi/4$	0°

magnitude Spectrum



phase response



phase and group delays

- In some applications, time delay of ~~the~~ a signal passing Through a filter is more significant the phase. so This delay is known as phase delay $\boxed{T_p(\omega) = \frac{-\theta(\omega)}{\omega}}$

→ when i/p contain many frequencies that are not harmonically related is processed by a filter, each has different delays(phase) we group Them Then delay is group delay $\tau_g = \frac{-d\theta(\omega)}{d\omega}$